WELL-POSEDNESS WITHOUT SEMICONTINUITY: FROM PARAMETRIC QUASIEQUILIBRIA TO OPTIMIZATION WITH EQUILIBRIUM CONSTRAINTS

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Abstract. We consider optimization-related problems, from parametric quasiequilibrium and quasioptimization problems to parametric equilibrium problems with equilibrium constraints and finally optimization problems with equilibrium constraints. We propose a relaxed sequential level closedness and use it together with sequential pseudocontinuity assumptions to establish sufficient conditions for parametric well-posedness and well-posedness. For topological settings we use sensitivity analysis while for problems on metric spaces we argue on diameters and Kuratowski's and Hausdorff's measures of noncompactness of approximate solution sets. Besides some new results we also improve or generalize several recent ones in the literature.

Keywords. Quasiequilibria, quasioptimization, equilibrium constraints, (unique) well-posedness, parametric well-posedness, sequential level closedness, sequential pseudocontinuity

Mathematics Subject Classifications 49K27, 49K40, 90C31.

1 Introduction.

In their seminal papers, Hadamard [13] and Tikhonov [30] initiated two ways of developing well-posedness study for various mathematical problems. For constrained optimization the pioneer work was [20] of Levitin and Polyak, who extended the definition for unconstrained problems in [30]. Observe that the notions of Hadamard and Tikhonov were proved closely related in [7, 29]. Recently, these two notions have been more blended and linked to stability theory in parametric well-posedness study [5, 17, 18, 27, 31, 32, 34]. Well-posedness for various problems related to

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optimization has been recently intensively considered, see e.g.: for optimization problems in [14-16, 27, 29, 33, 34], for variational inequalities in [8, 9, 11, 21, 24], for Nash equilibria in [23, 25], for fixed-point problems in [11, 18, 19], for inclusion problems in [11, 18, 19], for equilibrium problems in [5, 12, 17] and for bilevel problems in [5, 12, 17, 22, 23]. In most cases it is commonly assumed at least that the involved functions are sequentially lower semicontinuous. In [26, 27] a weaker notion of sequential lower pseudocontinuity is introduced to investigate parametric constrained optimization. In this paper we propose generalized sequential level closedness definitions and use them together with sequential pseudocontinuity to consider well-posedness in the Tikhonov sense, which is more important in approximation study and numerical algorithms, because all algorithms consist of providing sequences of approximate solutions convergent to an exact one. Simple examples (e.g. Examples 2.1 and 2.2) ensure that these properties are properly weaker than semicontinuity and hence results under assumptions about these properties are significant in practical situations. We choose to investigate rather general optimization-related problems to include a wide range of particular cases. Namely we begin with parametric quasiequilibrium and quasioptimization problems. Note that quasiequilibrium models contain quasivariational inequalities, complementarity problems, vector minimization problems, Nash equilibria, fixed-point and coincidence-point problems, traffic networks, etc. A quasioptimization problem is more general than an optimization one as constraint sets depend on the decision variable as well. This is a special case of a quasiequilibrium problem but we go into details due to its importance. Then we pass to bilevel models to discuss first equilibrium problems with equilibrium constraints. Of course, bilevel considerations are more general than "single level" ones. Finally we investigate optimization problems with equilibrium constraints, which have been recently intensively investigated in the literature. We discuss well-posedness by tools of sensitivity analysis for general settings in topological spaces, since this property is closely related to stability, especially for parametric problems. When decision spaces are metric spaces, diameters and measures of noncompactness of approximate solution sets play a crucial role. Namely, well-posedness depends on whether these quantities tend to zero or not. We will be employing both Kuratowski's and Hausdorff's measures in this paper. Furthermore, in our results for optimization problems, a kind of marginal functions participates as well. Since the solution existence of these problems have been intensively studied, we focus on well-posedness assuming always that solutions of the problem under consideration exist. Some of our results improve the counterparts in the recent papers [12, 22, 27]. The others are new.

In the rest of this section we state our problems and recall well-posedness notions. Section 2 is devoted to generalized sequential level closedness and sequential pseudocontinuity properties. In the next section 3 we establish sufficient conditions for a quasiequilibrium problem to be parametrically well-posed. Section 4 contains well-posedness conditions for a quasioptimization problem. Parametric well-posedness of equilibrium problems with equilibrium constraints is the subject of section 5. In the last section 6 we discuss well-posedness of optimization problems with equilibrium constraints.

Let X and Λ be Hausdorff topological spaces, $f : X \times X \times \Lambda \to R$ and $K_i : X \times \Lambda \to 2^X$, $i = 1, 2$. Our parametric equilibrium problem consists of, for each $\lambda \in \Lambda$,

$$(\mathrm{QEP}_\lambda) \qquad \text{finding } \bar{x} \in K_1(\bar{x}, \lambda) \text{ such that, for all } y \in K_2(\bar{x}, \lambda),$$

$$f(\bar{x}, y, \lambda) \geq 0.$$

Instead of writing $\{(\mathrm{QEP}_\lambda) : \lambda \in \Lambda\}$ for the family of problems, i.e. the parametric problem, we will simply write (QEP) in the sequel.

Let X and Λ be Hausdorff topological spaces, $g : X \times \Lambda \to \bar{R}$, where $\bar{R} = (-\infty, +\infty]$, and $K : X \times \Lambda \to 2^X$. Our parametric quasioptimization problem is, for each $\lambda \in \Lambda$,

$$(\mathrm{QOP}_\lambda) \qquad \begin{cases} \text{minimize} \quad g(x, \lambda) \\ \text{subject to } x \in K(x, \lambda). \end{cases}$$

Similarly as for (QEP), we denote simply (QOP) for this family of quasioptimization problems.

Let X and $\mathit{\Lambda}$ be Hausdorff topological spaces. Let $S : \mathit{\Lambda} \to X$ be the solution map of the parametric quasiequilibrium problem (QEP). Let $Y = X \times \mathit{\Lambda}$ and $F : Y \times Y \to R$. The parametric equilibrium problem with equilibrium constraints we consider is of

(EPEC) finding $\bar{\boldsymbol{y}} \in \mathrm{gr}S$ such that, for all $\boldsymbol{y} \in \mathrm{gr}S$,

$$F(\bar{\boldsymbol{y}}, \boldsymbol{y}) \geq 0.$$

Let $S : \Lambda \to X$ be the solution map of parametric quasiequilibrium problem (QEP). The optimization problem with equilibrium constraints under question is

$$(\mathrm{OPEC}) \qquad \begin{cases} \text{minimize} \quad g(x, \lambda) \\ \text{subject to } \boldsymbol{x} := (x, \lambda) \in \mathrm{gr}S, \end{cases}$$

where $\mathrm{gr}S$ denotes the graph of S, i.e. $\mathrm{gr}S := \{(x, \lambda) : x \in S(\lambda)\}$. Note that although λ is the parameter of the quasiequilibrium problem defining the constraint, it is a component of the decision variable (x, λ) of (OPEC) and this problem is not parametric.

We first recall well-posedness notions

Definition 1.1 Let $\{\lambda_n\}$ converge to $\bar{\lambda}$. For $x_n \in K_1(x_n, \lambda_n)$, the sequence $\{x_n\}$ is said to be an approximating sequence for (QEP) corresponding to $\{\lambda_n\}$, if there exists a sequence $\{\varepsilon_n\}$ convergent to 0^+ such that, for all $y \in K_2(x_n, \lambda_n)$,

$$f(x_n, y, \lambda_n) + \epsilon_n \geq 0.$$

Definition 1.2 Problem (QEP) is called well-posed at $\bar{\lambda}$ if

(a) the solution set $S(\bar{\lambda})$ of $(\mathrm{QEP}_{\bar{\lambda}})$ is nonempty;

(b) for any sequence $\{\lambda_n\}$ convergent to $\bar{\lambda}$, every corresponding approximating sequence for (QEP) has a subsequence convergent to some point of $S(\bar{\lambda})$.

(QEP) is called uniquely well-posed at $\bar{\lambda}$ if $S(\bar{\lambda}) = \{\bar{x}\}$, a singleton, and every approximating sequence converges to $\bar{x}$. (QEP) (or any other problem) is called parametrically (uniquely) well-posed if it is (uniquely) well-posed at each $\lambda \in \Lambda$.

Definition 1.3 Let $\{\lambda_n\}$ converge to $\bar{\lambda}$ in Λ. For $x_n \in K(x_n, \lambda_n)$, the sequence $\{x_n\}$ is said to be an approximating (or minimizing) sequence for (QOP) corresponding to $\{\lambda_n\}$, if there exists a sequence $\{\varepsilon_n\} \subseteq (0, +\infty)$ convergent to 0 such that

$$g(x_n, \lambda_n) \leq \inf_{x \in K(x_n, \lambda_n)} g(x, \lambda_n) + \varepsilon_n.$$

Definition 1.4 Problem (QOP) is called well-posed at $\bar{\lambda}$ if

(a) $(\text{QOP}_{\bar{\lambda}})$ has solutions;

(b) for any sequence $\{\lambda_n\}$ convergent to $\bar{\lambda}$, every corresponding approximating sequence for (QOP) has a subsequence convergent to some point of $S(\bar{\lambda})$.

We say that (QOP) is uniquely well-posed at $\bar{\lambda}$ if $S(\bar{\lambda}) = \{\bar{x}\}$, a singleton, and every approximating sequence converges to $\bar{x}$.

Definition 1.5 A sequence $\{\boldsymbol{y}_n\} := \{(x_n, \lambda_n)\} \subseteq X \times \mathit{\Lambda}$ is termed an approximating sequence for (EPEC) if there exists $\{\varepsilon_n\} \to 0^+$ such that

(i) $F(\boldsymbol{y}_n, \boldsymbol{y}) + \varepsilon_n \geq 0,$ for all $y \in S(\lambda)$ and all $\lambda \in \mathit{\Lambda},$ where $\boldsymbol{y} := (y, \lambda)$;

(ii) $\{x_n\}$ is an approximating sequence for the parametric problem (QEP) corresponding to $\{\lambda_n\}$.

Definition 1.6 (EPEC) is said to be well-posed if

(i) it has at least one solution;

(ii) every approximating sequence for (EPEC) has a subsequence convergent to a solution.

Furthermore, we say that (EPEC) is uniquely well-posed if it has a unique solution and any approximating sequence converges to this solution.

Definition 1.7 A sequence $\{\boldsymbol{y}_n\} := \{(x_n, \lambda_n)\} \subseteq X \times \mathit{\Lambda}$ is called an approximating (or minimizing) sequence for (OPEC) if there exists $\{\varepsilon_n\} \to 0^+$ such that

(i) $g(\boldsymbol{y}_n) \leq g(\boldsymbol{y}) + \varepsilon_n,$ for all $\lambda \in \mathit{\Lambda}$ and all $y \in S(\lambda)$, where $\boldsymbol{y} := (y, \lambda)$;

(ii) $\{x_n\}$ is an approximating sequence for (QEP) corresponding to $\{\lambda_n\}$.

Definition 1.8 Problem (OPEC) is called well-posed if

(i) it has solutions;

(ii) every approximating sequence for (OPEC) has a subsequence convergent to a solution.

(OPEC) is termed uniquely well-posed if it has a unique solution and every approximating sequence converges to this solution.

Note that, in the above definitions, like a number of authors, we require an approximating sequence to be (strictly) included in the constraint set, unlike the definition in [20].

2 Generalized level closedness and pseudocontinuity of functions.

Let X be a topological space, $x_0 \in X$ and $f : X \to \bar{R}$. Recall that f is called sequentially upper (lower, respectively) semicontinuous, written shortly as usc (lsc, resp), at x_0 if, for all sequences $\{x_n\}$ convergent to x_0, $f(x_0) \geq \limsup f(x_n)$ ($f(x_0) \leq \liminf f(x_n)$, resp). Note that in this paper we are concerned always with sequential properties. Hence we write clearly "sequential" or "sequentially" only to remind the reader in case necessary. Observe that f is usc at x_0 if and only if for all $\{x_n\} \to x_0$ and all $b \in R$,

$$[f(x_n) \geq b, \forall n] \Rightarrow [f(x_0) \geq b]$$

and similarly for lower semicontinuity. Therefore, we propose the following natural definition.

Definition 2.1 Let X and Y be topological spaces, $f : X \to \bar{R}$ and $g : Y \to \bar{R}$.

(i) f is called (sequentially) upper 0-level closed with respect to (wrt) g at $(x_0, y_0) \in X \times Y$ if, for any sequence $\{(x_n, y_n)\}$ convergent to (x_0, y_0),

$$[f(x_n) + g(y_n) \geq 0, \forall n] \Rightarrow [f(x_0) + g(y_0) \geq 0].$$

(ii) f is called (sequentially) lower 0-level closed wrt g at (x_0, y_0) if, for any sequence $\{(x_n, y_n)\}$ convergent to (x_0, y_0),

$$[f(x_n) + g(y_n) \leq 0, \forall n] \Rightarrow [f(x_0) + g(y_0) \leq 0].$$

If we have f in place of $f + g$ in the above inequalities, we say that f is upper (or lower) 0-level closed at x_0. While if we have $b \in R$ instead of 0, then of course "0-level" is replaced by "b-level"

Remark 2.1 If f and g are usc (lsc, resp) at x_0 and y_0, respectively, then f is upper (lower, resp) 0-level closed wrt g at (x_0, y_0). Indeed, if $\{(x_n, y_n)\} \to (x_0, y_0)$ and $f(x_n) + g(y_n) \geq 0$ for all n, one has

$$f(x_0) + g(y_0) \geq \limsup f(x_n) + \limsup g(y_n) \geq \limsup[f(x_n) + g(y_n)] \geq 0.$$

From now on we use id to denote the identity map on R_+. The following example shows that the converse of the above remark is not true.

Example 2.1 Let $f : R \to R$ be defined by

$$f(x) = \begin{cases} 0, & \text{if } x \in Q, \\ 1, & \text{if } x \in R \setminus Q, \end{cases}$$

where Q is the set of the rational numbers. Then f is upper 0-level closed wrt id at (x, y), for all $(x, y) \in R \times R_+$, but f is neither usc at any $x \in Q$ nor lsc at any $x \in R \setminus Q$.

DEFINITION 2.2 [26, 27] Let X be a topological space and $f : X \to \bar{R}$.

(a) f is said to be (sequentially) upper pseudocontinuous at $x_0 \in X$ if,

$$[f(x) > f(x_0)] \Rightarrow [\text{for any } \{x_n\} \to x_0, f(x) > \limsup f(x_n)].$$

(b) f is called lower pseudocontinuous at $x_0 \in X$ if,

$$[f(x) < f(x_0)] \Rightarrow [\text{for any } \{x_n\} \to x_0, f(x) < \liminf f(x_n)].$$

(c) f is termed pseudocontinuous at $x_0 \in X$ if it is both lower and upper pseudocontinuous at this point.

The class of the upper pseudocontinuous functions strictly contains that of the usc functions, see [26]. We include here a very simple illustrative example.

Example 2.2 Let $f : R \to R$ be defined by

$$f(x) = \begin{cases} x + 1, & \text{if } \ x > 0, \\ 0, & \text{if } \ x = 0, \\ x - 1, & \text{if } \ x < 0. \end{cases}$$

Then, f is pseudocontinuous at 0 but neither usc nor lsc at 0.

We note further that if f and g are lsc (or usc) at x_0 then $f + g$ is lsc (usc, resp) at x_0. Unfortunately, this property does not hold for pseudocontinuous functions as shown by

Example 2.3 Let $f_1, g_1 : R \to R$ be defined as follows

$$f_1(x) = \begin{cases} 1, & \text{if } \ x \geq 0, \\ \frac{x}{2}, & \text{if } \ x < 0 \end{cases} \quad \text{and} \quad g_1(x) = -x.$$

Then, f_1 is lower pseudocontinuous at 0 and g_1 is continuous at 0. But

$$(f_1 + g_1)(x) = \begin{cases} -x + 1, & \text{if } \ x \geq 0, \\ -\frac{x}{2}, & \text{if } \ x < 0. \end{cases}$$

is not lower pseudocontinuous at 0.

To see the same situation for upper pseudocontinuity let

$$f_2(x) = \begin{cases} -1, & \text{if } \ x \geq 0, \\ -\frac{x}{2}, & \text{if } \ x < 0 \end{cases} \qquad \text{and} \quad g_2(x) = x.$$

Then at 0, f_2 is upper pseudocontinuous and g_2 is continuous. However,

$$(f_2 + g_2)(x) = \begin{cases} x - 1, & \text{if } \ x \geq 0, \\ \frac{x}{2}, & \text{if } \ x < 0. \end{cases}$$

is not upper pseudocontinuous at 0.

LEMMA 2.1 ([27], Proposition 2.3) *Let X be a topological space. Then $f : X \to \bar{R}$ is pseudocontinuous in X if and only if, for all sequences $\{x_n\}$ and $\{y_n\}$ in X, convergent to x and y, respectively,*

$$[f(y) < f(x)] \Rightarrow [\limsup f(y_n) < \liminf f(x_n)].$$

3 Quasiequilibrium problem (QEP).

We consider first parametric problem (QEP) stated in section 1. For well-posedness in general topological settings we need the following facts which are well-known and often used in sensitivity analysis (see e.g. [1-4] and references therein).

Remark 3.1 Let $Q : X \to 2^Y$ be a multimap between two topological spaces. Then the following assertions hold.

(i) If $Q(\bar{x})$ is compact, then Q is usc at $\bar{x}$ if and only if for any sequence $\{x_n\}$ convergent to $\bar{x}$ and $y_n \in Q(x_n)$, there is a subsequence $\{y_{n_k}\}$ convergent to some $y \in Q(\bar{x})$.

(ii) If, in addition, $Q(\bar{x}) = \{\bar{y}\}$ is a singleton then the above limit point y must be $\bar{y}$ and the whole $\{y_n\}$ converges to $\bar{y}$.

By $S(\lambda)$ we denote the solution set of (QEP$_\lambda$). For positive ε, the ε-solution set of (QEP$_\lambda$) is defined by

$$\widetilde{S}(\lambda, \varepsilon) = \{x \in K_1(x, \lambda) \mid f(x, y, \lambda) + \varepsilon \geq 0, \forall y \in K_2(x, \lambda)\}.$$

When X and Λ are metric spaces, for positive ζ and ε, we define the following set of approximate solutions of the family (QEP), allowing also the parametric to vary around the considered point,

$$\Pi(\bar{\lambda}, \zeta, \varepsilon) := \bigcup_{\lambda \in B(\bar{\lambda}, \zeta)} \widetilde{S}(\lambda, \varepsilon),$$

where $B(\bar{\lambda}, \zeta)$ is the closed ball centered at $\bar{\lambda}$ and with radius ζ.

THEOREM 3.1 *Assume that*

(i) *X is compact, K_1 is closed and K_2 is lsc in $X \times \{\bar{\lambda}\}$;*

(ii) f *is upper* 0*-level closed wrt* id *in* $K_1(X,\bar{\lambda}) \times K_2(X,\bar{\lambda}) \times \{\bar{\lambda}\} \times \{0\}$.

Then (QEP) *is well-posed at* $\bar{\lambda}$. *Furthermore, if* $S(\bar{\lambda})$ *is a singleton, then this problem is uniquely well-posed at* $\bar{\lambda}$.

Proof. We first check that $\widetilde{S}(.,.)$ is usc at $(\bar{\lambda}, 0)$. Suppose to the contrary the existence of an open superset U of $\widetilde{S}(\bar{\lambda}, 0)$ such that for all $\{(\lambda_n, \varepsilon_n)\}$ convergent to $(\bar{\lambda}, 0)$ in $\Lambda \times R_+$, there is $x_n \in \widetilde{S}(\lambda_n, \varepsilon_n)$ such that $x_n \notin U$, for all n. By the compactness of X one can assume that $\{x_n\}$ converges to some x_0. Since K_1 is closed at $(x_0, \bar{\lambda})$, $x_0 \in K_1(x_0, \bar{\lambda})$. If $x_0 \notin \widetilde{S}(\bar{\lambda}, 0) = S(\bar{\lambda})$, there is $y_0 \in K_2(x_0, \bar{\lambda})$ such that $f(x_0, y_0, \bar{\lambda}) < 0$. The lower semicontinuity of K_2 in turn shows the existence of $y_n \in K_2(x_n, \lambda_n)$ such that $\{y_n\} \to y_0$. As $x_n \in \widetilde{S}(\lambda_n, \varepsilon_n)$, one has

$$f(x_n, y_n, \lambda_n) + \varepsilon_n \geq 0.$$

By the upper 0-level closedness wrt id of f, we have

$$f(x_0, y_0, \lambda) \geq 0,$$

which is a contradiction. Thus, $x_0 \in \widetilde{S}(\bar{\lambda}, 0) \subseteq U$, which is another contradiction, since $x_n \notin U$, for all n. Hence, $\widetilde{S}$ is usc at $(\bar{\lambda}, 0)$.

Now we prove that $S(\bar{\lambda})$ is compact by checking its closedness. Let $x_n \in S(\bar{\lambda})$ converge to x_0. If $x_0 \notin S(\bar{\lambda})$, there exists $y_0 \in K_2(x_0, \bar{\lambda})$ such that

$$f(x_0, y_0, \bar{\lambda}) < 0.$$

In light of the lower semicontinuity of K_2 there is $y_n \in K_2(x_n, \bar{\lambda})$ such that $\{y_n\} \to y_0$. For all n one has

$$f(x_n, y_n, \bar{\lambda}) \geq 0$$

as $x_n \in S(\bar{\lambda})$. By assumption (ii), one has

$$f(x_0, y_0, \bar{\lambda}) \geq 0,$$

which is impossible. Therefore, $x_0 \in S(\bar{\lambda})$ and hence $S(\bar{\lambda})$ is compact. By Remark 3.1 we are done. □

The assumptions of Theorem 3.1 are essential as indicated in the following examples.

Example 3.1 (the compactness of X cannot be dropped). Let $X = R, \Lambda = R_+$, $K_1(x, \lambda) = K_2(x, \lambda) = R$, $\bar{\lambda} = 0$ and $f(x, y, \lambda) = 2^{x-y} + \lambda$. It is clear that in $X \times \Lambda$, K_1 is closed and K_2 is lsc. (ii) holds as f is continuous in $X \times X \times \Lambda$. But $S(\lambda) = R$ for all $\lambda \in \Lambda$. Hence, (QEP) is not well-posed at 0. Indeed, let $\lambda_n = \frac{1}{n} \to 0$ and $x_n = n \in S(\bar{\lambda}_n)$ for all n. It is clear that $\{x_n\}$ has no convergent subsequence. The reason is that X is not compact.

Example 3.2 (the closedness of K_1 is essential). Let $X = [-2,1], \Lambda = [0,1]$, $K_1(x,\lambda) = (-2\lambda, 1]$, $K_2(x,\lambda) = [0,1]$, $\bar{\lambda} = 0$ and $f(x,y,\lambda) = x(x-y)$. It is not hard to see that X is compact, K_2 is lsc in $X \times \Lambda$, (ii) is fulfilled (by the continuity of f). But $S(0) = \{1\}$ and $S(\lambda) = \{0,1\}$ for all $\lambda \in (0,1]$. Therefore, (QEP) is not well-posed at 0. The reason is that K_1 is not closed at $X \times \{0\}$. Indeed, let $x_n = \lambda_n = \frac{1}{n}$ and $z_n = -\frac{1}{n} \in K_1(x_n,\lambda_n) = (-\frac{2}{n}, 1]$. We see that $\{z_n\}$ tends to $0 \notin K_1(0,0)$.

Example 3.3 (the lower semicontinuity of K_2 cannot be dispensed). Let $X = [-1,1], \Lambda = [0,1]$. $K_1(x,\lambda) = [0,1]$, $f(x,y,\lambda) = x+y$, $\bar{\lambda} = 0$ and

$$K_2(x,\lambda) = \begin{cases} \{-1,0,1\}, & \text{if} \quad \lambda = 0, \\ \{0,1\}, & \text{otherwise.} \end{cases}$$

Then X is compact, K_1 is closed in $X \times \Lambda$ and (ii) holds (by the continuity of f in $X \times X \times \Lambda$). But $S(0) = \{1\}$ and $S(\lambda) = \{0,1\}$ for all $\lambda \in (0,1]$. Thus, (QEP) is not well-posed at 0. The reason is that K_2 is not lsc in $X \times \{\bar{\lambda}\}$.

Example 3.4 ((ii) cannot be dropped). Let $X = [0,1], \Lambda = [0,1], K_1(x,\lambda) \equiv K_2(x,\lambda) = [0,1]$ and

$$f(x,y,\lambda) = \begin{cases} x-y, & \text{if} \quad \lambda = 0, \\ y-x, & \text{otherwise.} \end{cases}$$

It is clear that assumption (i) is satisfied and $S(0) = \{1\}$. Let $\lambda_n = \varepsilon_n = \frac{1}{n}$, and $x_n = 0 \in \widetilde{S}(\lambda_n,\varepsilon_n)$. Then $\{x_n\}$ is an approximating sequence for (QEP) corresponding to $\{\lambda_n\}$. But $\{x_n\} \to 0 \notin S(0)$ and hence $\{(\mathrm{QEP}_\lambda) : \lambda \in \Lambda\}$ is not well-posed at $\bar{\lambda} = 0$. The reason is that assumption (ii) is violated. Indeed, taking $x_n = 0, y_n = 1, \lambda_n = \frac{1}{n}$ and $\varepsilon_n = 0$, we have $\{(x_n,y_n,\lambda_n,\varepsilon_n)\} \to (0,1,0,0)$ and $f(x_n,y_n,\lambda_n) + \varepsilon_n = f(0,1,\frac{1}{n}) = 1 > 0$ but $f(0,1,0) = -1 < 0$.

Remark 3.2 In the special case where $K(x,\lambda) \equiv X$, it is not hard to check that the assumption (ii) for f can be reduced to the same condition for $f(.,y,.)$, for all $y \in X$. Therefore, Theorem 3.1 improves Theorem 3.3 in [12]. Indeed, it suffices to check assumption (ii) of Theorem 3.1 from the (assumed in [12]) monotonicity of $f(.,.,\bar{\lambda})$ and lower semicontinuity of $f(x,.,.)$. If $\{(x_n,\lambda_n)\} \to (x,\bar{\lambda})$ and $\{\varepsilon_n\}$ tends to 0^+ are such that

$$f(x_n,y,\bar{\lambda}_n) + \varepsilon_n \geq 0,$$

then, by the monotonicity, the inequalities

$$f(y,x,\bar{\lambda}) \leq \liminf f(y,x_n,\lambda_n) \leq \liminf f(x_n,y,\lambda_n) \leq \liminf \varepsilon_n = 0$$

imply that $f(x,y,\bar{\lambda}) \geq 0$. Note further that we omit the hemicontinuity of $f(.,.,\bar{\lambda})$ and convexity of $f(x,.,\bar{\lambda})$ imposed in [12].

THEOREM 3.2 *Let X and Λ be metric spaces.*

(i) *If* (QEP) *is uniquely well-posed at* $\bar{\lambda}$*, then* $\mathrm{diam}\Pi(\bar{\lambda},\zeta,\varepsilon)\to 0^+$ *as* $(\zeta,\varepsilon)\to(0^+,0^+)$.

(ii) *Conversely, if* X *is complete and the following conditions hold*

(a) K_1 *is closed and* K_2 *is lsc in* $X\times\{\bar{\lambda}\}$;

(ii) f *is upper* 0*-level closed wrt* id *in* $K_1(X,\bar{\lambda})\times K_2(X,\bar{\lambda})\times\{\bar{\lambda}\}\times\{0\}$,

then (QEP) *is uniquely well-posed at* $\bar{\lambda}$*, provided that* $\mathrm{diam}\Pi(\bar{\lambda},\zeta,\varepsilon)\to 0^+$ *as* $(\zeta,\varepsilon)\to(0^+,0^+)$.

Proof. (i) Suppose (QEP) is uniquely well-posed at $\bar{\lambda}$, but there is $\{(\zeta_n,\varepsilon_n)\}\to(0^+,0^+)$ such that there are $n_0\in\mathcal{N}$ (the set of the natural number) and $r>0$ such that, for all $n\geq n_0$,

$$\mathrm{diam}\Pi(\bar{\lambda},\zeta_n,\varepsilon_n)>r.$$

Then, there exist $x_n^1,x_n^2\in\Pi(\bar{\lambda},\zeta_n,\varepsilon_n)$ such that $d(x_n^1,x_n^2)>\frac{r}{2}$. Consequently, there are $\lambda_n^1,\lambda_n^2\in B(\bar{\lambda},\zeta_n)$ such that

$$f(x_n^1,y,\lambda_n^1)+\varepsilon_n\geq 0,\forall y\in K(x_n^1,\lambda_n^1)$$

and

$$f(x_n^2,y,\lambda_n^2)+\varepsilon_n\geq 0,\forall y\in K(x_n^2,\lambda_n^2),$$

i.e. $\{x_n^1\}$ and $\{x_n^2\}$ are approximating sequences for (QEP) corresponding to $\{\lambda_n^1\}$ and $\{\lambda_n^2\}$, respectively. Hence, $\{x_n^1\}$ and $\{x_n^2\}$ converge to the unique solution of $(\mathrm{QEP}_{\bar{\lambda}})$, contradicting the fact that $d(x_n^1,x_n^2)>\frac{r}{2}>0$, for all n.

(ii) Let $\{\lambda_n\}\to\bar{\lambda}$ and $\{x_n\}$ be an approximating sequence for (QEP) corresponding to $\{\lambda_n\}$. Then there is $\{\varepsilon_n\}\to 0^+$ such that, for all $y\in K_2(x_n,\lambda_n)$ and all $n\in\mathcal{N}$,

$$f(x_n,y,\lambda_n)+\varepsilon_n\geq 0.$$

Consequently, x_n belongs to $\Pi(\bar{\lambda},\zeta_n,\varepsilon_n)$ with $\{\zeta_n\}:=\{d(\lambda_n,\bar{\lambda})\}\to 0^+$ as $n\to+\infty$. Since $\mathrm{diam}\Pi(\bar{\lambda},\zeta_n,\varepsilon_n)\to 0^+$, $\{x_n\}$ is a Cauchy sequence and converges to some $\bar{x}$. By the closedness of K_1 at $(\bar{x},\bar{\lambda})$, $\bar{x}\in K(\bar{x},\bar{\lambda})$. Using the same argument as for Theorem 3.1, we deduce that $\bar{x}\in S(\bar{\lambda})$. To complete the proof one shows that $(\mathrm{QEP}_{\bar{\lambda}})$ has a unique solution. If $S(\bar{\lambda})$ has two distinct solutions $\bar{x}_1$ and $\bar{x}_2$, it is not hard to see that $\bar{x}_1$ and $\bar{x}_2$ belong to $\Pi(\bar{\lambda},\zeta,\varepsilon)$, for all positive ζ and ε. It follows that

$$0<d(\bar{x}_1,\bar{x}_2)\leq\mathrm{diam}\Pi(\bar{\lambda},\zeta,\varepsilon),$$

which is impossible. □

Remark 3.3 If $K(x,\lambda)\equiv X$, with the same argument as in Remark 3.2, we see that Theorem 3.2 improves Theorem 3.1 of [12]. Here we omit the hemicontinuity of $f(.,.,\bar{\lambda})$ and convexity of $f(x,.,\bar{\lambda})$, which are required in that theorem.

The following example shows that we cannot replace the assumed unique well-posedness in Theorem 3.2 (i) by well-posedness.

Example 3.5 Let $X = \Lambda = [0,1], K_1(x,\lambda) \equiv K_2(x,\lambda) = [0,1]$ and $f(x,y,\lambda) = 1$. Then (QEP) is well-posed in Λ. But $\Pi(\lambda,\zeta,\varepsilon) = [0,1]$ and hence its diameter does not converge to 0.

Now we need the following notions of measures of noncompactness.

DEFINITION 3.1 Let M be a nonempty subset of a metric space X.

(i) The Kuratowski measure of M is

$$\mu(M) = \inf\{\varepsilon > 0 \mid M \subseteq \bigcup_{k=1}^{n} M_k \text{ and } \mathrm{diam} M_k \leq \varepsilon, k = 1, ..., n, \text{ for some } n \in \mathcal{N}\}.$$

(ii) The Hausdorff measure of M is

$$\eta(M) = \inf\{\varepsilon > 0 \mid M \subseteq \bigcup_{k=1}^{n} B(x_k,\varepsilon), x_k \in X, \text{ for some } n \in \mathcal{N}\}.$$

The following inequalities are obtained in [10]

$$\eta(M) \leq \mu(M) \leq 2\eta(M).$$

The measures μ and η share many properties and we will use γ in the sequel to denote either one of them. γ is a regular measure (see [6, 28]), i.e. it enjoys the following properties

(a) $\gamma(M) = +\infty$ if and only if the set M is unbounded;

(b) $\gamma(M) = \gamma(\mathrm{cl}M)$;

(c) from $\gamma(M) = 0$ it follows that M is a totally bounded set;

(d) if X is a complete space and if $\{A_n\}$ is a sequence of closed subsets of X such that $A_{n+1} \subseteq A_n$ for each $n \in \mathcal{N}$ and $\lim_{n\to+\infty} \gamma(A_n) = 0$, then $K := \bigcap_{n\in\mathcal{N}} A_n$ is a nonempty compact set and $\lim_{n\to+\infty} H(A_n, K) = 0^+$, where H is the Hausdorff metric;

(e) from $M \subseteq N$ it follows that $\gamma(M) \leq \gamma(N)$.

THEOREM 3.3

(i) *If* (QEP) *is well-posed at* $\bar{\lambda}$, *then* $\gamma(\Pi(\bar{\lambda},\zeta,\varepsilon)) \to 0^+$ *as* $(\zeta,\varepsilon) \to (0^+,0^+)$.

(ii) *Conversely, if* X *is complete,* Λ *is compact or finite dimensional and the following conditions hold*

  (a) K_1 *is closed and* K_2 *is lsc in* $X \times \Lambda$;

  (b) f *is upper b-level closed in* $K_1(X,\Lambda) \times K_2(X,\Lambda) \times \Lambda$, *for all* $b < 0$,

then (QEP) *is well-posed at* $\bar{\lambda}$, *provided that* $\gamma(\Pi(\bar{\lambda},\zeta,\varepsilon)) \to 0^+$ *as* $(\zeta,\varepsilon) \to (0^+,0^+)$.

Proof. Let γ be the Hausdorff measure η (for the Kuratowski measure case the argument is similar).

(i) Assume that (QEP) is well-posed at $\bar{\lambda}$ and $(\zeta, \varepsilon) \to (0^+, 0^+)$. Since $S(\bar{\lambda}) \subseteq \Pi(\bar{\lambda}, \zeta, \varepsilon)$ for all $\zeta, \varepsilon > 0$,

$$H(\Pi(\bar{\lambda}, \zeta, \varepsilon)), S(\bar{\lambda})) = H^*(\Pi(\bar{\lambda}, \zeta, \varepsilon), S(\bar{\lambda})),$$

where $H^*(A, B) = \sup_{a \in A} d(a, B)$ and $d(a, B) = \inf_{b \in B} d(a, b)$. Let $\{x_n\}$ be any sequence in $S(\bar{\lambda})$. Since $\{x_n\}$ is an approximating sequence for (QEP), there is subsequence convergent to some point of $S(\bar{\lambda})$. Hence, $S(\bar{\lambda})$ is compact.

If $S(\bar{\lambda}) \subseteq \bigcup_{k=1}^{n} B(z_k, \varepsilon)$, then

$$\Pi(\bar{\lambda}, \zeta, \varepsilon) \subseteq \bigcup_{k=1}^{n} B\big(z_k, \varepsilon + H\big(\Pi(\bar{\lambda}, \zeta, \varepsilon), S(\bar{\lambda})\big)\big)$$

and hence

$$\eta(\Pi(\bar{\lambda}, \zeta, \varepsilon)) \leq H\big(\Pi(\bar{\lambda}, \zeta, \varepsilon), S(\bar{\lambda})\big) + \gamma(S(\bar{\lambda}))\big).$$

Since $S(\bar{\lambda})$ is compact, $\eta(S(\bar{\lambda})) = 0$. So we have

$$\eta(\Pi(\bar{\lambda}, \zeta, \varepsilon)) \leq H(\Pi(\bar{\lambda}, \zeta, \varepsilon), S(\bar{\lambda})).$$

Now we claim that $H(\Pi(\bar{\lambda}, \zeta, \varepsilon), S(\bar{\lambda})) \to 0^+ \quad \text{as} \quad (\zeta, \varepsilon) \to (0^+, 0^+)$. Indeed, suppose to the contrary that there are $\rho > 0$, $\{(\zeta_n, \varepsilon_n)\} \to (0^+, 0^+)$ and $x_n \in \Pi(\bar{\lambda}, \zeta_n, \varepsilon_n)$ such that, for all $n \in \mathcal{N}$,

$$d(x_n, S(\bar{\lambda})) \geq \rho.$$

Since $\{x_n\}$ is an approximating sequence for (QEP), there is a subsequence convergent to some point of $S(\bar{\lambda})$, a contradiction.

(ii) Assume that $\eta(\Pi(\bar{\lambda}, \zeta, \varepsilon)) \to 0^+ \quad \text{as} \quad (\zeta, \varepsilon) \to (0^+, 0^+)$. We first prove that $\Pi(\bar{\lambda}, \zeta, \varepsilon)$ is closed for all positive ζ and ε. Let $x_n \in \Pi(\bar{\lambda}, \zeta, \varepsilon)$ be such that $\{x_n\} \to x$. Then, for each $n \in \mathcal{N}$, there is $\lambda_n \in B(\bar{\lambda}, \zeta)$ such that, for all $y \in K_2(x_n, \lambda_n)$,

$$f(x_n, y, \lambda_n) + \varepsilon \geq 0.$$

Since $B(\bar{\lambda}, \zeta)$ is compact, we can assume that $\{\lambda_n\} \to \lambda$ for some $\lambda \in B(\bar{\lambda}, \zeta)$. By the closedness of K_1 at (x, λ), $x \in K_1(x, \lambda)$. We claim that, for all $y \in K_2(x, \lambda)$,

$$f(x, y, \lambda) + \varepsilon \geq 0.$$

Indeed, if there exists $y \in K_2(x, \lambda)$ such that $f(x, y, \lambda) + \varepsilon < 0$, there is $y_n \in K_2(x_n, \lambda_n)$ such that $\{y_n\} \to y$ as K_2 is lsc at (x, λ). By the upper $-\varepsilon$-level closedness of f at (x, y, λ), there is $n_0 \in \mathcal{N}$ such that, for all $n \geq n_0$,

$$f(x_n, y_n, \lambda_n) < -\varepsilon,$$

a contradiction. Since $\lambda \in B(\bar{\lambda}, \zeta)$, we have $x \in \Pi(\bar{\lambda}, \zeta, \varepsilon)$. Hence, $\Pi(\bar{\lambda}, \zeta, \varepsilon)$ is closed.

Now we show that $S(\bar{\lambda}) = \bigcap_{\zeta>0,\varepsilon>0} \Pi(\bar{\lambda}, \zeta, \varepsilon)$. We first check that $\bigcap_{\zeta>0} \Pi(\bar{\lambda}, \zeta, \varepsilon) = \widetilde{S}(\bar{\lambda}, \varepsilon)$. Indeed, it is easy to see that $\bigcap_{\zeta>0} \Pi(\bar{\lambda}, \zeta, \varepsilon) \supseteq \widetilde{S}(\bar{\lambda}, \varepsilon)$. Let $x \in \bigcap_{\zeta>0} \Pi(\bar{\lambda}, \zeta, \varepsilon)$. There is $\lambda_n \in B(\bar{\lambda}, \zeta)$ such that, for all $y \in K_2(x, \lambda_n)$, $f(x, y, \lambda_n) + \varepsilon \geq 0$. Since $x \in K_1(x, \lambda_n)$, $\{\lambda_n\} \to \bar{\lambda}$ and K_1 is closed, one sees that $x \in K_1(x, \bar{\lambda})$. Now we verify that $x \in \widetilde{S}(\bar{\lambda}, \varepsilon)$. Indeed, for each $y \in K_2(x, \bar{\lambda})$, since K_2 is lsc at $(x, \bar{\lambda})$, there exists $y_n \in K_2(x, \lambda_n)$ with $\{y_n\} \to y$. Since $x \in \widetilde{S}(\lambda_n, \varepsilon)$,

$$f(x, y_n, \lambda_n) + \varepsilon \geq 0.$$

By the upper $-\varepsilon$-level closedness of f, one has

$$f(x, y, \bar{\lambda}) + \varepsilon \geq 0,$$

i.e. $\bigcap_{\zeta>0} \Pi(\bar{\lambda}, \zeta, \varepsilon) \subseteq \widetilde{S}(\bar{\lambda}, \varepsilon)$. Hence, $\bigcap_{\zeta>0} \Pi(\bar{\lambda}, \zeta, \varepsilon) = \widetilde{S}(\bar{\lambda}, \varepsilon)$. Next, we have $S(\bar{\lambda}) = \bigcap_{\varepsilon>0} \widetilde{S}(\bar{\lambda}, \varepsilon) = \bigcap_{\zeta>0,\varepsilon>0} \Pi(\bar{\lambda}, \zeta, \varepsilon)$.

Since $\eta(\Pi(\bar{\lambda}, \zeta, \varepsilon)) \to 0^+$ as $(\zeta, \varepsilon) \to (0^+, 0^+)$, the regular measure properties of η imply that $S(\bar{\lambda})$ is compact and $H(\Pi(\bar{\lambda}, \zeta, \varepsilon), S(\bar{\lambda})) \to 0^+ \quad \text{as} \quad (\zeta, \varepsilon) \to (0^+, 0^+)$.

Let x_n be an approximating sequence for (QEP) corresponding to $\{\lambda_n\}$, where $\{\lambda_n\} \to \bar{\lambda}$. There is $\{\varepsilon_n\} \to 0^+$ such that, for all $y \in K_2(x_n, \lambda_n)$ and all $n \in \mathcal{N}$,

$$f(x_n, y, \lambda_n) + \varepsilon_n \geq 0.$$

This means that $x_n \in \Pi(\bar{\lambda}, \zeta_n, \varepsilon_n)$ with $\zeta_n := d(\bar{\lambda}, \lambda_n)$. We see that

$$d(x_n, S(\bar{\lambda})) \leq H(\Pi(\bar{\lambda}, \zeta_n, \varepsilon_n), S(\bar{\lambda})) \to 0^+.$$

Hence, there is $\bar{x}_n \in S(\bar{\lambda})$ such that

$$d(x_n, \bar{x}_n) \to 0 \quad \text{as} \quad n \to \infty.$$

By the compactness of $S(\bar{\lambda})$, there is a subsequence $\{\bar{x}_{n_k}\}$ of $\{\bar{x}_n\}$ convergent to some point $\bar{x}$ of $S(\bar{\lambda})$. Therefore, the corresponding subsequence $\{x_{n_k}\}$ of $\{x_n\}$ tends to $\bar{x}$. Hence, (QEP) is well-posed at $\bar{\lambda}$. $\square$

The following examples show that the assumptions of Theorem 3.3 (ii) are essential.

Example 3.6 Let $X = R, \mathit{\Lambda} = [0, 1]$, $K_1(x, \lambda) = (-\lambda, 1]$, $K_2(x, \lambda) \equiv [0, 1]$, $f(x, y, \lambda) = x(x - y)$ and $\bar{\lambda} = 0$. It is easy to see that X is complete, $\mathit{\Lambda}$ is compact, K_2 is lsc in $X \times \mathit{\Lambda}$. Condition (ii b) holds since f is continuous in $X \times X \times \mathit{\Lambda}$. Moreover, $\Pi(0, \zeta, \varepsilon) \subseteq [-1, 1]$ and hence $\gamma(\Pi(0, \zeta, \varepsilon)) = 0$. But $S(0) = \{1\}$ and $S(\lambda) = \{0, 1\}$ for all $\lambda \in (0, 1]$. Hence, (QEP) is not well-posed at 0. The reason is that K_1 is not closed at $(0, 0)$. Indeed, let $x_n = \lambda_n = \frac{1}{n}$ and $z_n = \frac{1}{n} \in K_1(x_n, \lambda_n)$. We see that $z_n \to 0 \notin K_1(0, 0)$, and hence K_1 is not closed at $(0, 0)$.

Example 3.7 Let X, Λ and $\bar{\lambda}$ be as in Example 3.6, $K_1(x,\lambda) = [0,1]$, $f(x,y,\lambda) = x + y$ and

$$K_2(x,\lambda) = \begin{cases} \{-1,0,1\}, & \text{if} \quad \lambda = 0, \\ \{0,1\}, & \text{otherwise.} \end{cases}$$

It is not hard to see that X is complete, Λ is compact, K_1 is closed in $X \times \Lambda$. (ii b) is satified as f is continuous in $X \times X \times \Lambda$. $\Pi(0,\zeta,\varepsilon) \subseteq [0,1]$ and hence $\gamma(\Pi(0,\zeta,\varepsilon)) = 0$. But $S(0) = \{1\}$, $S(\lambda) = \{0,1\}$ for all $\lambda \in (0,1]$. Thus, (QEP) is not well-posed at 0. The reason is that K_2 is not lsc in $X \times \Lambda$.

Example 3.8 Let $X, \Lambda, K_1, \bar{\lambda}$ be as in Example 3.7, $K_2(x,\lambda) = \{\lambda, 1+\lambda\}$ and

$$f(x,y,\lambda) = \begin{cases} -1, & \text{if} \quad x + y = 1, \\ 1, & \text{otherwise.} \end{cases}$$

It is clear that X is complete, Λ is compact, (ii a) holds and $\gamma(\Pi(0,\zeta,\varepsilon)) = 0$. But $S(0) = (0,1)$, $S(\lambda) = [0,1]$ for all $\lambda \in (0,1]$. Therefore, (QEP) is not well-posed at 0. The reason is that assumption (iib) is violated. Indeed, let $(x_n, y_n, \lambda_n) = (\frac{1}{n}, 1 - 2\frac{1}{n}, \frac{1}{n})$. We see that

$$f(x_n, y_n, \lambda_n) = 1 \geq -\frac{1}{2}.$$

But $\{(x_n, y_n, \lambda_n)\} \to (0,1,0)$ and

$$f(0,1,0) = -1 \ngeq -\frac{1}{2}.$$

Remark 3.4 In the special case where $K(x,\lambda) \equiv X$, it is easy to see that assumption (iib) of Theorem 3.3 can be reduced to the corresponding one of $f(.,y,\lambda)$, for all $y \in X$. Theorem 3.2 of [12] has the same conclusion as Theorem 3.3, but only for the Kuratowski measure μ. Observe that the upper semicontinuity of $f(.,y,.)$, required in that theorem, implies the upper b-level closednees of $f(.,y,.)$ for all $b < 0$ as imposed in Theorem 3.3. Note further that (see Proposition 2.1 of [5]) the upper semicontinuity of $f(.,y,.)$ is equivalent to the upper b-level closedness of $f(.,y,.)$ for all b.

The following example gives a case where Theorem 3.3 is easy to be employed, but Theorem 3.2 of [12] does not work.

Example 3.9 Let $X = \Lambda = [0,1]$, $K_1(x,\lambda) = K_2(x,\lambda) = [0,1]$, $\bar{\lambda} = 0$ and

$$f(x,y,\lambda) = \begin{cases} 0, & \text{if} \quad \lambda \in [0,1] \cap Q, \\ 1, & \text{if} \quad \lambda \in [0,1] \cap (R \setminus Q). \end{cases}$$

Then the assumptions in (ii) of Theorem 3.3 are satified, and hence this theorem yields the well-posedness of (QEP) at 0. (In fact, $S(\lambda) = [0,1]$ for all $\lambda \in [0,1]$.) But $f(.,y,.)$ is not usc in $X \times \Lambda$, and hence Theorem 3.2 of [12] is not in use.

4 Quasioptimization Problem (QOP).

We first investigate parametric well-posedness of this problem in topological settings.

Theorem 4.1 *Assume that*

(i) *X is compact and K is closed and lsc in $X \times \{\bar{\lambda}\}$;*

(ii) *g is pseudocontinuous in $K(X, \bar{\lambda}) \times \{\bar{\lambda}\}$.*

Then (QOP) *is well-posed at $\bar{\lambda}$. Furthermore, if* (QOP) *has a unique solution, this problem is uniquely well-posed at $\bar{\lambda}$.*

Proof. By setting $K_1(x, \lambda) = K_2(x, \lambda) = K(x, \lambda)$, for all $(x, \lambda) \in X \times \Lambda$ and $f(x, y, \lambda) = g(y, \lambda) - g(x, \lambda)$, (QOP) becomes a special case of (QEP). To apply Theorem 3.1 we check its assumption (ii). Let x_n and y_n be in $K(X, \lambda_n)$ and $\varepsilon_n \in (0, +\infty)$ be such that $\{(x_n, y_n, \lambda_n, \varepsilon_n)\} \to (x, y, \bar{\lambda}, 0)$ and

$$f(x_n, y_n, \lambda_n) + \varepsilon_n \geq 0.$$

There are $\bar{x}_n$ and $\bar{y}_n$ in X such that $x_n \in K(\bar{x}_n, \lambda_n)$ and $y_n \in K(\bar{y}_n, \lambda_n)$. Due to the compactness of X one can assume that $\{\bar{x}_n\} \to \bar{x}$ and $\{\bar{y}_n\} \to \bar{y}$, for some $\bar{x}, \bar{y} \in X$. As K is closed in $X \times \{\bar{\lambda}\}$, we have $\bar{x} \in K(\bar{x}, \bar{\lambda})$ and $\bar{y} \in K(\bar{y}, \bar{\lambda})$.

Now suppose ad absurdum that

$$g(y, \bar{\lambda}) < g(x, \bar{\lambda}).$$

By Lemma 2.1 we have

$$\limsup g(y_n, \lambda_n) < \liminf g(x_n, \lambda_n).$$

Hence, there are $t_1, t_2 \in R$ and $n_0 \in \mathcal{N}$ such that, for $n \geq n_0$,

$$g(y_n, \lambda_n) \leq t_1 < t_2 \leq g(x_n, \lambda_n)$$

and then

$$g(y_n, \lambda_n) - g(x_n, \lambda_n) \leq t_1 - t_2 < 0,$$

which is impossible and we are done. □

Let $m : X \times \Lambda \to R$ be the following kind of marginal functions

$$m(x, \lambda) := \inf\{g(y, \lambda) \mid y \in K(x, \lambda)\}.$$

When (QOP) is given on metric spaces, similarly as for (QEP) we define $\widetilde{S}$ and Π as follows

$$\widetilde{S}(\lambda, \varepsilon) = \{x \in K(x, \lambda) \mid g(x, \lambda) \leq m(x, \lambda) + \varepsilon\},$$

$$\Pi(\bar{\lambda}, \zeta, \varepsilon) = \bigcup_{\lambda \in B(\bar{\lambda}, \zeta)} \widetilde{S}(\lambda, \varepsilon).$$

Theorem 4.2 *Assume that*

(i) *X is compact and K is closed in $X \times \{\bar{\lambda}\}$;*

(ii) *g is lower pseudocontinuous in $K(X, \bar{\lambda}) \times \{\bar{\lambda}\}$;*

(iii) *m is usc in $K(X, \bar{\lambda}) \times \{\bar{\lambda}\}$.*

Then (QOP) *is well-posed at $\bar{\lambda}$. Furthermore, if* (QOP) *has a unique solution, it is uniquely well-posed at $\bar{\lambda}$.*

Proof. We check first that $\widetilde{S}$ is usc at $(\bar{\lambda}, 0)$. Suppose to the contrary the existence of an open superset U of $\widetilde{S}(\bar{\lambda}, 0)$ such that for all $\{(\lambda_n, \varepsilon_n)\}$ convergent to $(\bar{\lambda}, 0^+)$ in $\Lambda \times R_+$, there is $x_n \in \widetilde{S}(\lambda_n, \varepsilon_n)$ such that $x_n \notin U$, for all n. By the compactness of X one can assume that $\{x_n\}$ tends to some x_0. Since K is closed at $(x_0, \bar{\lambda})$, $x_0 \in K(x_0, \bar{\lambda})$. If $x_0 \notin \widetilde{S}(\bar{\lambda}, 0) = S(\bar{\lambda})$, there is $y_0 \in K(x_0, \bar{\lambda})$ such that

$$g(y_0, \bar{\lambda}) < g(x_0, \bar{\lambda}).$$

Since g is lower pseudocontinuous at $(x_0, \bar{\lambda})$, we have

$$m(x_0, \bar{\lambda}) \leq g(y_0, \bar{\lambda}) < \liminf g(x_n, \lambda_n).$$

The upper semicontinuity of m at $(x_0, \bar{\lambda})$ yields some $t \in R$ such that

$$\limsup m(x_n, \lambda_n) < t < \liminf g(x_n, \lambda_n).$$

Hence, there is $n_0 \in \mathcal{N}$ such that, for all $n \geq n_0$,

$$m(x_n, \lambda_n) - g(x_n, \lambda_n) < t - g(x_n, \lambda_n).$$

As $x_n \in \widetilde{S}(\lambda_n, \varepsilon_n)$,

$$-\varepsilon_n \leq m(x_n, \lambda_n) - g(x_n, \lambda_n) \leq 0.$$

Therefore,

$$0 = \lim_{n \to +\infty} [m(x_n, \lambda_n) - g(x_n \lambda_n)] \leq t - \liminf_{n \to +\infty} g(x_n, \lambda_n) < 0.$$

This contradiction shows that $x_0 \in S(\bar{\lambda})$. Then another contradiction is obtained as $x_n \notin U$. Thus, $\widetilde{S}$ is usc at $(\bar{\lambda}, 0)$. Now we prove that $S(\bar{\lambda})$ is compact by checking its closedness. Let $\{x_n\} \subseteq S(\bar{\lambda})$ converge to x_0. As $S(\bar{\lambda}) \subseteq \widetilde{S}(\bar{\lambda}, \varepsilon_n)$, by the preceding argument one sees that $x_0 \in S(\bar{\lambda})$. By Remark 3.1, (QOP) is well-posed at $\bar{\lambda}$. □

The following examples explain that Theorems 4.1 and 4.2 are incomparable and each of them may be applicable in different situations.

Example 4.1 Let $X = \Lambda = [0, 1], K(x, \lambda) = [0, 1]$, $\bar{\lambda} = 1$ and

$$g(x, \lambda) = \begin{cases} (1 + x)(1 - \lambda), & \text{if} \quad 0 \leq \lambda < 1, \\ -1, & \text{if} \quad \lambda = 1. \end{cases}$$

It is clear that K is continuous, X is compact and g is lower pseudocontinuous in $[0,1]\times[0,1]$. Now we check that g is upper pseudocontinuous at $(x,1)$, for all $x\in[0,1]$. Indeed, assume that $g(y,\lambda)>g(x,1)=-1$ and $\{(x_n,\lambda_n)\}\to(x,1)$. It is clear that, $g(y,\lambda)>0$ as $\lambda<1$ and $\limsup_{n\to+\infty} g(x_n,\lambda_n)=0$, so $g(y,\lambda)>\limsup_{n\to+\infty} g(x_n,\lambda_n)$. Hence, the assumptions of Theorem 4.1 are satisfied and we obtain the well-posedness at 1 (in fact, $S(1)=[0,1]$ and $S(\lambda)=\{0\}$ for all $0\le\lambda<1$). However,

$$m(x,\lambda)\equiv m(\lambda)=\begin{cases}1-\lambda, & \text{if } \ 0\le\lambda<1,\\ -1, & \text{if } \ \lambda=1\end{cases}$$

is not usc at 1. Therefore, Theorem 4.2 cannot be applied in this case.

Example 4.2 Let $X=\Lambda=[0,1], K(x,\lambda)=[0,1]$, $\bar\lambda=0$ and

$$g(x,\lambda)=\begin{cases}0, & \text{if } \ \lambda=0 \text{ and } 0\le x<1,\\ \lambda(1-x), & \text{if } \ 0<\lambda\le1 \text{ and } 0\le x<1,\\ -1, & \text{if } \ x=1.\end{cases}$$

Then K is continuous and X is compact. g is lower pseudocontinuous at $(x,0)$, for all $x\in[0,1]$. Indeed, if $g(y,\lambda)<g(x,0)$ then $x<1$, and hence $g(x,0)=0$. So $g(y,\lambda)=-1$ and $y=1$. If $\{(x_n,\lambda_n)\}\to(x,0)$, there is $n_0\in\mathcal{N}$ such that, for all $n\ge n_0$, $x_n<1$. So, we have $\liminf g(x_n,\lambda_n)=0$. Thus, $g(y,\lambda)<\liminf g(x_n,\lambda_n)$, i.e, g is lower pseudocontinuous at $(x,0)$. However, g is not upper pseudocontinuous in $[0,1]\times\{0\}$. Indeed, let $y=\frac12$ and $\lambda=0$. Then

$$0=g(\frac12,0)>g(1,0)=-1.$$

Take $x_n=1-\frac{1}{n+1}$ and $\lambda_n=\frac{1}{n+1}$. Then $\{(x_n,\lambda_n)\}\to(1,0)$ as $n\to+\infty$. It is easy to see that

$$\limsup g(x_n,\lambda_n)=\limsup \lambda_n(1-x_n)=0,$$

and hence $g(\frac12,0)\not>\limsup g(x_n,\lambda_n))$. Therefore, Theorem 4.1 is not in use. Fortunately, the assumptions of Theorem 4.2 are satisfied, since $m(x,\lambda)\equiv m(\lambda)=\inf_{x\in[0,1]} g(x,\lambda)=-1$, for all $\lambda\in[0,1]$ and hence m is continuous in $[0,1]$. Theorem 4.2 yields the well-posedness of (QOP) at 0 (in fact, $S(\lambda)=\{1\}$, for all $\lambda\in[0,1]$).

Now we pass to well-posedness conditions in terms of the diameter of $\Pi(\lambda,\zeta,\varepsilon)$.

THEOREM 4.3 *Assume that X is a metric space.*

(i) *If* (QOP) *is uniquely well-posed at $\bar\lambda$, then* $\text{diam}\Pi(\bar\lambda,\zeta,\varepsilon)\to0^+$ *as* $(\zeta,\varepsilon)\to(0^+,0^+)$, *where* diam(.) *denotes the diameter of the set (.).*

(ii) *Conversely, assume that X is complete and the following conditions hold*

(a) *K is closed and lsc in $X\times\{\bar\lambda\}$;*

(b) *either of the following two conditions holds* (b$_1$) *g is pseudocontinuous in $K(X,\bar\lambda)\times\{\bar\lambda\}$;*

(b$_2$) *in $K(X,\bar\lambda)\times\{\bar\lambda\}$, g is lower pseudocontinuous and m is usc.*

Then (QOP) *is uniquely well-posed at $\bar\lambda$, provided that* $\text{diam}\Pi(\bar\lambda,\zeta,\varepsilon)\to 0^+$ *as* $(\zeta,\varepsilon)\to(0^+,0^+)$.

Proof. (i) Suppose (QOP) is uniquely well-posed at $\bar\lambda$ but, for $\{(\zeta_n,\varepsilon_n)\}\to(0^+,0^+)$, there are $n_0\in\mathcal{N}$ and $r>0$ such that, for all $n\ge n_0$,

$$\text{diam}\Pi(\bar\lambda,\zeta_n,\varepsilon_n)>r.$$

Then, there exist $x_n^1,x_n^2\in\Pi(\bar\lambda,\zeta_n,\varepsilon_n)$ such that $d(x_n^1,x_n^2)>\frac{r}{2}$. There are $\lambda_n^1,\lambda_n^2\in B(\bar\lambda,\zeta_n)$ such that

$$g(x_n^1,\lambda_n^1)\le m(x_n^1,\lambda_n^1)+\varepsilon_n$$

and

$$g(x_n^2,\lambda_n^2)\le m(x_n^2,\lambda_n^2)+\varepsilon_n.$$

Since $\{x_n^1\}$ and $\{x_n^2\}$ are approximating sequences for (QOP) corresponding to $\{\lambda_n^1\}$ and $\{\lambda_n^2\}$, respectively, they converge to the unique solution and we obtain a contradiction.

(ii) Assume that $\{\lambda_n\}\to\bar\lambda$ and $\{x_n\}$ is an approximating sequence for (QOP) corresponding to $\{\lambda_n\}$. Then, there is $\{\varepsilon_n\}\to 0^+$ such that, for all $n\in\mathcal{N}$,

$$g(x_n,\lambda_n)\le m(x_n\lambda_n)+\varepsilon_n.$$

Hence x_n belongs to $\Pi(\bar\lambda,\zeta_n,\varepsilon_n)$ with $\zeta_n:=d(\lambda_n,\bar\lambda)$. Since $\lim_{n\to+\infty}\text{diam}\Pi(\bar\lambda,\zeta_n,\varepsilon_n)=0^+$, $\{x_n\}$ is a Cauchy sequence and hence converges to some $\bar x$. The closedness of K_1 implies that $\bar x\in K(\bar x,\bar\lambda)$. Using the same argument as for Theorem 4.1 for the case (b$_1$) or Theorem 4.2 for the case (b$_2$), we see that $\bar x\in S(\bar\lambda)$. To complete the proof, we have to show that (QOP$_{\bar\lambda}$) has a unique solution. If $S(\bar\lambda)$ has two distinct solutions $\bar x_1$ and $\bar x_2$, they clearly belong to $\Pi(\bar\lambda,\zeta,\varepsilon)$, for all positive ζ and ε. This implies the contradiction that

$$0<d(\bar x_1,\bar x_2)\le\text{diam}\Pi(\bar\lambda,\zeta,\varepsilon).\quad\square$$

Theorem 4.4

(i) $\gamma(\Pi(\bar\lambda,\zeta,\varepsilon))\to 0^+$ *as* $(\zeta,\varepsilon)\to(0^+,0^+)$, *if* (QOP) *is well-posed at $\bar\lambda$ (recall that γ is the Kuratowski measure or Hausdorff measure).*

(ii) *Conversely, assume that X is complete and Λ is compact or finite dimensional. Impose further that,*

(a) *K is closed in $X\times\Lambda$;*

(b) *g is lsc in $K(X,\Lambda)\times\Lambda$;*

(c) *m is usc in $K(X,\Lambda)\times\Lambda$.*

Then (QOP) *is well-posed at* $\bar{\lambda}$*, provided that* $\gamma(\Pi(\bar{\lambda},\zeta,\varepsilon)) \to 0^+$ *as* $(\zeta,\varepsilon) \to (0^+,0^+)$.

Proof. By the similarity we discuss only the case where $\gamma = \mu$, the Kuratowski measure.

(i) Assume that (QOP) is well-posed at $\bar{\lambda}$. Since, for all positive ζ and ε, $S(\bar{\lambda}) \subseteq \Pi(\bar{\lambda},\zeta,\varepsilon)$, one has

$$H(\Pi(\bar{\lambda},\zeta,\varepsilon)), S(\bar{\lambda})) = H^*(\Pi(\bar{\lambda},\zeta,\varepsilon), S(\bar{\lambda})).$$

Let $\{x_n\}$ be a sequence in $S(\bar{\lambda})$. Then $\{x_n\}$ is an approximating sequence for (QOP) and has a subsequence convergent to some point of $S(\bar{\lambda})$. Hence, $S(\bar{\lambda})$ is compact.

Let $S(\bar{\lambda}) \subseteq \bigcup_{k=1}^{n} M_k$ with $\text{diam}M_k \leq \varepsilon$, for $k = 1, ..., n$. Setting

$$N_k = \{z \in X \mid d(z, M_k) \leq H(\Pi(\bar{\lambda},\zeta,\varepsilon), S(\bar{\lambda}))\}.$$

We claim that

$$\Pi(\bar{\lambda},\zeta,\varepsilon) \subseteq \bigcup_{k=1}^{n} N_k.$$

Indeed, let $x \in \Pi(\bar{\lambda},\zeta,\varepsilon)$. Then $d(x, S(\bar{\lambda})) \leq H(\Pi(\bar{\lambda},\zeta,\varepsilon), S(\bar{\lambda}))$. Since $S(\bar{\lambda}) \subseteq \bigcup_{k=1}^{n} M_k$, we see that $d(x, \bigcup_{k=1}^{n} M_k) \leq H(\Pi(\bar{\lambda},\zeta,\varepsilon), S(\bar{\lambda}))$. Hence, there is $\bar{k}$ such that $d(x, M_{\bar{k}}) \leq H(\Pi(\bar{\lambda},\zeta,\varepsilon), S(\bar{\lambda}))$, i.e. $x \in N_{\bar{k}}$. So, $\Pi(\bar{\lambda},\zeta,\varepsilon) \subseteq \bigcup_{k=1}^{n} N_k$. Note further that

$$\text{diam}N_k = \text{diam}M_k + 2H\big(\Pi(\bar{\lambda},\zeta,\varepsilon), S(\bar{\lambda})\big) \leq \varepsilon + 2H\big(\Pi(\bar{\lambda},\zeta,\varepsilon), S(\bar{\lambda})\big),$$

and hence, as $\mu(S(\bar{\lambda})) = 0$,

$$\mu(\Pi(\bar{\lambda},\zeta,\varepsilon)) \leq 2H\big(\Pi(\bar{\lambda},\zeta,\varepsilon), S(\bar{\lambda})\big) + \mu(S(\bar{\lambda})) = 2H(\Pi(\bar{\lambda},\zeta,\varepsilon), S(\bar{\lambda})).$$

Now we prove that $H(\Pi(\bar{\lambda},\zeta,\varepsilon), S(\bar{\lambda})) \to 0^+ \quad \text{as} \quad (\zeta,\varepsilon) \to (0^+,0^+)$. Suppose to the contrary that there are $\rho > 0$, $\{(\zeta_n,\varepsilon_n))\} \to (0^+,0^+)$ and $x_n \in \Pi(\bar{\lambda},\zeta_n,\varepsilon_n)$ such that, for all $n \in \mathcal{N}$,

$$d(x_n, S(\bar{\lambda})) \geq \rho.$$

Since $\{x_n\}$ is an approximating sequence for (QOP), it has a subsequence convergent to some point of $S(\bar{\lambda})$, a contradiction. Therefore, $\mu(\Pi(\bar{\lambda},\zeta,\varepsilon)) \to 0^+$ as $(\zeta,\varepsilon) \to (0^+,0^+)$.

(ii) Assume that $\mu(\Pi(\bar{\lambda},\zeta,\varepsilon)) \to 0^+ \quad \text{as} \quad (\zeta,\varepsilon) \to (0^+,0^+)$. We first show that $\Pi(\bar{\lambda},\zeta,\varepsilon)$ is closed for all positive ζ and ε. Let $x_n \in \Pi(\bar{\lambda},\zeta,\varepsilon)$ and $\{x_n\} \to x$. Then, for each $n \in \mathcal{N}$, there is $\lambda_n \in B(\bar{\lambda},\zeta)$ such that

$$g(x_n,\lambda_n) \leq m(x_n,\lambda_n) + \varepsilon.$$

Because $B(\bar{\lambda},\zeta)$ is compact, we assume that $\{\lambda_n\} \to \lambda$ for some $\lambda \in B(\bar{\lambda},\zeta)$. Since K is closed at (x,λ), $x \in K(x,\lambda)$. By the lower semicontinuity of g and the upper semicontinuity of m at (x,λ), we have

$$g(x,\lambda) \leq m(x,\lambda) + \varepsilon.$$

As $\lambda \in B(\bar{\lambda}, \zeta)$ we have $x \in \Pi(\bar{\lambda}, \zeta, \varepsilon)$. Hence, $\Pi(\bar{\lambda}, \zeta, \varepsilon)$ is closed. Note further that $S(\bar{\lambda}) = \bigcap_{\zeta>0,\varepsilon>0} \Pi(\bar{\lambda}, \zeta, \varepsilon)$ and $\mu(\Pi(\bar{\lambda}, \zeta, \varepsilon)) \to 0^+$ as $(\zeta, \varepsilon) \to (0^+, 0^+)$. From the properties of μ it follows that $S(\bar{\lambda})$ is compact and

$$H(\Pi(\bar{\lambda}, \zeta, \varepsilon), S(\bar{\lambda})) \to 0^+.$$

Let $\{x_n\}$ be an approximating sequence for (QOP) corresponding to $\{\lambda_n\}$, where $\{\lambda_n\} \to \bar{\lambda}$. There is $\{\varepsilon_n\} \to 0^+$ such that, for all $n \in \mathcal{N}$,

$$g(x_n, \lambda_n) \leq m(x_n, \lambda_n) + \varepsilon_n.$$

Consequently, $x_n \in \Pi(\bar{\lambda}, \zeta_n, \varepsilon_n)$ with $\zeta_n := d(\bar{\lambda}, \lambda_n)$. We see that

$$d(x_n, S(\bar{\lambda})) \leq H(\Pi(\bar{\lambda}, \zeta_n, \varepsilon_n), S(\bar{\lambda})) \to 0^+.$$

By the compactness of $S(\bar{\lambda})$, there is a subsequence of $\{x_n\}$ converging to some point of $S(\bar{\lambda})$. Hence, (QOP) is well-posed at $\bar{\lambda}$. $\square$

THEOREM 4.5 *Assume that X is complete and Λ is compact or finite dimensional. Let the following conditions hold*

(a) *K is closed and lsc in $X \times \Lambda$;*

(b) *g is continuous in $K(X, \Lambda) \times \Lambda$.*

Then (QOP) *is well-posed at $\bar{\lambda}$, provided that $\gamma(\Pi(\bar{\lambda}, \zeta, \varepsilon)) \to 0^+$ as $(\zeta, \varepsilon) \to (0^+, 0^+)$.*

Proof. We consider only the case $\gamma = \mu$. Let $\mu(\Pi(\bar{\lambda}, \zeta, \varepsilon)) \to 0^+$ as $(\zeta, \varepsilon) \to (0^+, 0^+)$. We prove that $\Pi(\bar{\lambda}, \zeta, \varepsilon)$ is closed for all positive ζ and ε. Let $x_n \in \Pi(\bar{\lambda}, \zeta, \varepsilon)$ and $\{x_n\} \to x$. Then, for each $n \in \mathcal{N}$, there is $\lambda_n \in B(\bar{\lambda}, \zeta)$ such that

$$g(x_n, \lambda_n) \leq m(x_n, \lambda_n) + \varepsilon.$$

As $B(\bar{\lambda}, \zeta)$ is compact, we assume that $\{\lambda_n\} \to \lambda$ for some $\lambda \in B(\bar{\lambda}, \zeta)$. Then $x \in K(x, \lambda)$ as K is closed at (x, λ). Now we show that,

$$g(x, \lambda) \leq m(x, \lambda) + \varepsilon.$$

By the lower semicontinuity of g at (x, λ) we have

$$g(x, \lambda) \leq \liminf g(x_n, \lambda_n) \leq \liminf m(x_n, \lambda_n) + \varepsilon.$$

Hence, it is sufficient to check that

$$\liminf m(x_n, \lambda_n) \leq m(x, \lambda),$$

that is

$$\liminf \inf_{y \in K(x_n, \lambda_n)} g(y, \lambda_n) \leq \inf_{y \in K(x, \lambda)} g(y, \lambda).$$

Suppose to the contrary the existence of $\delta > 0$ such that

$$\liminf \inf_{y\in K(x_n,\lambda_n)} g(y,\lambda_n) = \inf_{y\in K(x,\lambda)} g(y,\lambda) + \delta.$$

Then, there is $y_0 \in K(x,\lambda)$ such that

$$\liminf \inf_{y\in K(x_n,\lambda_n)} g(y,\lambda_n) > g(y_0,\lambda) + \frac{\delta}{2}.$$

Since K is lsc at (x,λ), there is $y_n \in K(x_n,\lambda_n)$ such that $\{y_n\} \to y_0$. Taking into account the upper semicontinuity of g at (y_0,λ), one has

$$g(y_0,\lambda) \geq \limsup g(y_n,\lambda_n) \geq \liminf \inf_{y\in K(x_n,\lambda_n)} g(y,\lambda_n) > g(y_0,\lambda) + \frac{\delta}{2},$$

which is a contradiction. Therefore, as $\lambda \in B(\bar{\lambda},\zeta)$, we have $x \in \Pi(\bar{\lambda},\zeta,\varepsilon)$. Hence, $\Pi(\bar{\lambda},\zeta,\varepsilon)$ is closed. The further argument is the same as the last part of the proof of Theorem 4.4. □

Examples 4.1 and 4.2 show also that Theorems 4.4 and 4.5 are incomparable.

Remark 4.1 In the special case where $K(x,\lambda) \equiv K(\lambda)$, i.e. (QOP) becomes to an optimization problem, Theorems 4.1-4.3 collapse to Theorems 4.1-4.3 of [27]. Theorems 4.4 and 4.5 are new even for this special case.

5 Equilibrium problems with equilibrium constraints (EPEC).

This section is devoted to well-posedness conditions for (EPEC). For positive ζ and ε, the corresponding approximate solution set of (EPEC) is defined by

$$\Gamma(\zeta,\varepsilon) = \{\boldsymbol{x} = (x,\lambda) \in K_1(x,\lambda)\times\Lambda \mid F(\boldsymbol{x},\boldsymbol{y}) + \varepsilon \geq 0, \forall \boldsymbol{y} \in \mathrm{gr}S \text{ and}$$
$$f(x,y,\lambda) + \zeta \geq 0, \forall y \in K_2(x,\lambda)\}.$$

Theorem 5.1 *Assume that X is compact and*

(i) *in $X\times\Lambda$, K_1 is closed and K_2 is lsc;*

(ii) *f is upper 0-level closed wrt* id *in $K_1(X,\Lambda)\times K_2(X,\Lambda)\times\Lambda\times\{0\}$;*

(iii) *$F(.,\boldsymbol{y})$ is upper 0-level closed wrt* id *in $X\times\Lambda$, for all $\boldsymbol{y}\in X\times\Lambda$.*

Then (EPEC) is well-posed. Furthermore, if $S(\lambda)$ is a singleton for all $\lambda\in\Lambda$ and (EPEC) *admits a unique solution $\bar{\boldsymbol{x}}$, then* (EPEC) *is uniquely well-posed.*

Proof. Let $\{\boldsymbol{x}_n\} := \{(x_n,\lambda_n)\}$ be an approximating sequence for (EPEC). Assume that $\{\lambda_n\} \to \bar{\lambda}$. In light of Theorem 3.1, $\{(\mathrm{QEP}_\lambda) : \lambda\in\Lambda\}$ is parametrically well-posed. Since $\{x_n\}$ is an approximating sequence, there is a subsequence, still denoted by $\{x_n\}$, convergent to some

$\bar{x} \in S(\bar{\lambda})$. We show that $\bar{\boldsymbol{x}} := (\bar{x}, \bar{\lambda})$ is a solution of (EPEC). As $\{\boldsymbol{x}_n\}$ is an approximating sequence, there exists $\{\varepsilon_n\} \to 0^+$ such that, for all $\boldsymbol{y} \in$ grS,

$$F(\boldsymbol{x}_n, \boldsymbol{y}) + \varepsilon_n \geq 0.$$

The upper 0-level closedness of F implies that, for all $\boldsymbol{y} \in \mathrm{gr}S$,

$$F(\bar{\boldsymbol{x}}, \boldsymbol{y}) \geq 0,$$

i.e. $\bar{\boldsymbol{x}}$ is a solution. Thus, (EPEC) is well-posed at $\bar{\lambda}$.

To check the unique well-posedness under the additional conditions, let $\{\boldsymbol{x}_n\}$ be an approximating sequence for (EPEC). By the same argument as in the preceding part, there is a subsequence convergent to $\bar{\boldsymbol{x}}$. If $\{\boldsymbol{x}_n\}$ did not converge to $\bar{\boldsymbol{x}}$, there would be an open set $\mathcal{U}$ containing $\bar{\boldsymbol{x}}$ such that some subsequence was outside $\mathcal{U}$. By the above argument, this subsequence has a subsequence convergent to $\bar{\boldsymbol{x}}$, a contradiction. □

The assumptions of Theorem 5.1 cannot be dispensed as indicated in the following examples.

Example 5.1 (the compactness of X cannot be dropped). Let $X = R, \varLambda = [0,1]$, $K_1(x,\lambda) = K_2(x,\lambda) = R$, $f(x,y,\lambda) = 2^\lambda$ and $F((x,\lambda_1),(y,\lambda_2)) = 2^{x+y}$. It is clear that in $X \times \varLambda$, K_1 is closed and K_2 is lsc. (ii) and (iii) hold as f and F are continuous in $X \times X \times \varLambda$ and $(X \times \varLambda) \times (X \times \varLambda)$, respectively. Furthermore, the solutions set of (EPEC) is gr$S$. But $S(\lambda) = \mathcal{R}$ for all $\lambda \in \varLambda$, i.e. gr$S = \{(R,\lambda) \mid \lambda \in [0,1]\}$. Hence, (EPEC) is not well-posed. Indeed, let $x_n = n$, $\lambda_n = \frac{1}{n}$, $\boldsymbol{x_n} = (x_n, \lambda_n)$ is a solution of (EPEC). It is clear that $\{\boldsymbol{x}_n\}$ has no convergent subsequence. The reason is that X is not compact.

Example 5.2 (the closedness of K_1 is essential). Let $X = [-1,1], \varLambda = [0,1]$, $K_1(x,\lambda) = (-\lambda, 1]$, $K_2(x,\lambda) = [0,1]$, $f(x,y,\lambda) = x(x-y)$ and $F((x,\lambda_1),(y,\lambda_2)) = 1$. It is not hard to see that X is compact, K_2 is lsc in $X \times \varLambda$, (ii) and (iii) are satisfied (by the continuity of f and F). We see also that the solution set of (EPEC) is gr$S$. But $S(0) = \{1\}$ and $S(\lambda) = \{0,1\}$ for all $\lambda \in (0,1]$, i.e. gr$S = (1,0) \cup \{(k,\lambda) \mid k = 0,1; \lambda \in (0,1]\}$. Therefore, (EPEC) is not well-posed. Indeed, let $x_n = 0, \lambda_n = \frac{1}{n}$, then $\boldsymbol{x}_n = (x_n, \lambda_n)$ is a solution of (EPEC) and $\boldsymbol{x}_n$ converges to $\boldsymbol{x} = (0,1)$. But $\boldsymbol{x}$ does not belong to the solutions set of (EPEC). The reason is that K_1 is not closed at $X \times \varLambda$. Indeed, let $x_n = \lambda_n = \frac{1}{n}$ and $z_n = -\frac{1}{2n} \in K_1(x_n,\lambda_n) = (-\frac{1}{n}, 1]$. We see that $\{z_n\}$ tends to $0 \notin K_1(0,0)$.

Example 5.3 (the lower semicontinuity of K_2 cannot be dispensed). Let $X = [-1,1], \varLambda = [0,1]$, $K_1(x,\lambda) = [0,1]$, $f(x,y,\lambda) = x + y$, $F((x,\lambda_1),(y,\lambda_2)) = 2^{\lambda_1+\lambda_2}$ and

$$K_2(x,\lambda) = \begin{cases} \{-1,0,1\}, & \text{if } \lambda = 0, \\ \{0,1\}, & \text{otherwise.} \end{cases}$$

It is clear that X is compact, K_1 is closed in $X \times \varLambda$, (ii) and (iii) hold. Moreover, the solutions set of (EPEC) coincides with gr$S$. But $S(0) = \{1\}$, $S(\lambda) = \{0,1\}$ for all $\lambda \in (0,1]$, i.e.

gr$S = (1,0) \cup \{(k,\lambda) \mid k = 0,1; \lambda \in (0,1]\}$. By the same argument as in Example 5.2, (EPEC) is not well-posed. The reason is that K_2 is not lsc in $X \times \Lambda$.

Example 5.4 ((ii) cannot be dropped). Let $X = [0,1], \mathit{\Lambda} = [0,1], K_1(x,\lambda) \equiv K_2(x,\lambda) = [0,1]$, $F((x,\lambda_1),(y,\lambda_2)) = 0$ and

$$f(x,y,\lambda) = \begin{cases} x - y, & \text{if} \quad \lambda = 0, \\ y - x, & \text{otherwise.} \end{cases}$$

Then assumptions (i) and (iii) are satisfied and the solution set of (EPEC) is gr$S$. We see that $S(0) = \{1\}$ and $S(\lambda) = \{0\}$ for all $\lambda \in (0,1]$, and hence gr$S = (1,0) \cup \{(0,\lambda) \mid \lambda \in (0,1]\}$. By an argument similar to that in Example 5.2, we also see that (EPEC) is not well-posed. The reason is that assumption (ii) is violated. Indeed, taking $x_n = 0, y_n = 1, \lambda_n = \frac{1}{n}$ and $\varepsilon_n = 0$, we have $\{(x_n,y_n,\lambda_n,\varepsilon_n)\} \to (0,1,0,0)$ and $f(x_n,y_n,\lambda_n) + \varepsilon_n = f(0,1,\frac{1}{n}) = 1 > 0$ but $f(0,1,0) = -1 < 0$.

Example 5.5 ((iii) is essential). Let $X = [0,1], \mathit{\Lambda} = [0,1], K_1(x,\lambda) \equiv K_2(x,\lambda) = [0,1]$, $f(x,y,\lambda) = 1$ and

$$F((x,\lambda_1),(y,\lambda_2)) = \begin{cases} x - y, & \text{if} \quad \lambda_1 = 0, \\ y - x, & \text{otherwise.} \end{cases}$$

Then assumptions (i) and (ii) are satisfied and the solution set of (QEP) is equal to X. It is easy to see that the solution set of (EPEC) is the following subset of gr$S$

$$\boldsymbol{S} = (1,0) \cup \{(0,\lambda) \mid \lambda \in (0,1]\}.$$

By the same argument as in Example 5.2, we also see that (EPEC) is not well-posed. Similarly as in Example 5.4, we can check that assumption (iii) is violated.

Theorem 5.2 *Let X and Λ be metric spaces.*

(i) *If* (EPEC) *is uniquely well-posed at $\bar{\lambda}$, then* diam$\Gamma(\zeta,\varepsilon) \to 0^+$ *as* $(\zeta,\varepsilon) \to (0^+,0^+)$.

(ii) *Conversely, if X and Λ are complete and the following conditions hold*

  (a) *in $X \times \mathit{\Lambda}$, K_1 is closed and K_2 is lsc;*

  (b) *f is upper 0-level closed wrt* id *in $K_1(X,\mathit{\Lambda}) \times K_2(X,\mathit{\Lambda}) \times \mathit{\Lambda} \times \{0\}$;*

  (c) *$F(.,\boldsymbol{y})$ is upper 0-level closed wrt* id *in $X \times \mathit{\Lambda}$, for all $\boldsymbol{y} \in X \times \mathit{\Lambda}$,*

then (EPEC) *is uniquely well-posed at $\bar{\lambda}$, provided that* diam$\Gamma(\zeta,\varepsilon) \to 0^+$ *as* $(\zeta,\varepsilon) \to (0^+,0^+)$.

Proof. (i) Assume that (EPEC) is uniquely well-posed. Suppose to the contrary the existence of $\{(\zeta_n,\varepsilon_n)\} \to (0^+,0^+)$, $n_0 \in \mathcal{N}$ and $r > 0$ such that, for all $n \geq n_0$,

$$\text{diam}\Gamma(\zeta_n,\varepsilon_n) > r.$$

Hence, there exist $(x_n^1, \lambda_n^1), (x_n^2, \lambda_n^2) \in \Gamma(\zeta_n, \varepsilon_n)$ such that $d((x_n^1, \lambda_n^1), (x_n^2, \lambda_n^2)) > \frac{r}{2}$. Since $\{(x_n^1, \lambda_n^1)\}$ and $\{(x_n^2, \lambda_n^2)\}$ are approximating sequences, they converge to the unique solution and we obtain a contradiction.

(ii) Assume that $\{\boldsymbol{x}_n\} := \{(x_n, \lambda_n)\}$ is an approximating sequence for (EPEC). Then, $\boldsymbol{x}_n = (x_n, \lambda_n) \in \Gamma(\zeta_n, \varepsilon_n)$ and $\{\boldsymbol{x}_n\}$ is a Cauchy sequence and converges to some $\bar{\boldsymbol{x}} = (\bar{x}, \bar{\lambda})$. Since K_1 is closed at $(\bar{x}, \bar{\lambda})$ and $x_n \in K(x_n, \lambda_n)$, one has $\bar{x} \in K_1(\bar{x}, \bar{\lambda})$. Using the same argument as for Theorem 5.1, we see that $\bar{\boldsymbol{x}}$ solves (EPEC). We still have to prove that (EPEC) has a unique solution. Otherwise any pair of distinct solutions $(\bar{x}_1, \bar{\lambda}_1)$ and $(\bar{x}_2, \bar{\lambda}_2)$ belong to $\Gamma(\zeta, \varepsilon)$, for all positive ζ and ε. Then, we arrive at the contradiction that

$$0 < d((\bar{x}_1, \bar{\lambda}_1), (\bar{x}_2, \bar{\lambda}_2)) \leq \text{diam}\Gamma(\zeta, \varepsilon). \qquad \square$$

In terms of a measure $\gamma \in \{\mu, \eta\}$ of noncompactness we have the following result.

THEOREM 5.3 *Let X and Λ be metric spaces.*

(i) *If* (EPEC) *is well-posed then $\gamma(\Gamma(\zeta, \varepsilon)) \to 0^+$ as $(\zeta, \varepsilon) \to (0^+, 0^+)$.*

(ii) *Conversely, if X and Λ are complete and if the following conditions hold*

(a) *in $X \times \Lambda$, K_1 is closed and K_2 is lsc;*

(b) *f is upper upper b-level closed in $K_1(X, \Lambda) \times K_2(X, \Lambda) \times \Lambda$, for all $b < 0$;*

(c) *$F(., \boldsymbol{y})$ is upper c-level closed in $X \times \Lambda$, for all $\boldsymbol{y} \in X \times \Lambda$ and $c < 0$,*

then (EPEC) *is well-posed, provided that $\gamma(\Gamma(\bar{\lambda}, \zeta, \varepsilon)) \to 0^+$ as $(\zeta, \varepsilon) \to (0^+, 0^+)$.*

Proof. We discuss only the case $\gamma = \mu$, the Kuratowski measure.

(i) Assume that (EPEC) is well-posed. The solution set $\boldsymbol{S}$ of (EPEC) clearly satisfies the relation $\boldsymbol{S} \subseteq \Gamma(\zeta, \varepsilon)$. Hence,

$$H(\Gamma(\zeta, \varepsilon), \boldsymbol{S}) = H^*(\Gamma(\zeta, \varepsilon), \boldsymbol{S}).$$

Let $\{\boldsymbol{x}_n\} = \{(x_n, \lambda_n)\}$ be in $\boldsymbol{S}$. Since $\{\boldsymbol{x}_n\}$ is an approximating sequence, it has a subsequence convergent to some point of $\boldsymbol{S}$. Therefore, $\boldsymbol{S}$ is compact.

Assume that $\boldsymbol{S} \subseteq \bigcup_{k=1}^n M_k$ with $\text{diam}M_k \leq \varepsilon$, for $k = 1, ..., n$. Setting $N_k = \{z \in X \mid d(z, M_k) \leq H(\Gamma(\zeta, \varepsilon), \boldsymbol{S})\}$, it is easy to see that $\Gamma(\zeta, \varepsilon) \subseteq \bigcup_{k=1}^n N_k$ and $\text{diam}N_k \leq \varepsilon + 2H(\Gamma(\zeta, \varepsilon), \boldsymbol{S})$. Therefore,

$$\mu(\Gamma(\zeta, \varepsilon)) \leq 2H(\Gamma(\zeta, \varepsilon), \boldsymbol{S}) + \mu(\boldsymbol{S}) = 2H(\Gamma(\zeta, \varepsilon), \boldsymbol{S}).$$

To check that $H(\Gamma(\zeta, \varepsilon), \boldsymbol{S}) \to 0^+$ as $(\zeta, \varepsilon) \to (0^+, 0^+)$ by contradiction, suppose the existence of $\rho > 0$, $\{(\zeta_n, \varepsilon_n)\} \to (0^+, 0^+)$ and $\boldsymbol{x}_n \in \Gamma(\zeta_n, \varepsilon_n)$ such that, for all $n \in \mathcal{N}$,

$$d(\boldsymbol{x}_n, \boldsymbol{S}) \geq \rho.$$

Since $\{\boldsymbol{x}_n\}$ is an approximating sequence one has a subsequence convergent to some point of $\boldsymbol{S}$, which is impossible.

(ii) Assume that $\mu(\Gamma(\zeta,\varepsilon)) \to 0^+$ as $(\zeta,\varepsilon) \to (0^+,0^+)$. We claim that $\Gamma(\zeta,\varepsilon)$ is closed for all $\zeta,\varepsilon > 0$. Let $\boldsymbol{x}_n = (x_n,\lambda_n) \in \Gamma(\zeta,\varepsilon)$ with $\{\boldsymbol{x}_n\} \to \boldsymbol{x} := (x,\lambda)$. Then, for all $\boldsymbol{y} \in \mathrm{gr}S$ and all $y_n \in K_2(x_n,\lambda_n)$,

$$F(\boldsymbol{x}_n,\boldsymbol{y}) + \varepsilon \geq 0,$$

$$f(x_n,y,\lambda_n) + \zeta \geq 0.$$

As K_1 is closed at (x,λ), one has $x \in K_1(x,\lambda)$. By the upper $-\varepsilon$-level closedness of $F(.,\boldsymbol{y})$, one obtains, for all $\boldsymbol{y} \in \mathrm{gr}S$,

$$F(\boldsymbol{x},\boldsymbol{y}) + \varepsilon \geq 0.$$

Next we show by contraposition that, for all $y \in K_2(x,\lambda)$,

$$f(x,y,\lambda) + \zeta \geq 0.$$

Suppose there exists $y \in K_2(x,\lambda)$ such that $f(x,y,\lambda)+\zeta < 0$. Since K_2 is lsc at (x,λ), there is $y_n \in K_2(x_n,\lambda_n)$ such that $\{y_n\} \to y$. By the upper $-\zeta$-level closedness of f at (x,y,λ), there is $n_0 \in \mathcal{N}$ such that, for all $n \geq n_0$,

$$f(x_n,y_n,\lambda_n) < -\zeta,$$

a contradiction. As a result, $\boldsymbol{x} \in \Gamma(\zeta,\varepsilon)$ and this set is closed.

Note further that $\boldsymbol{S} = \bigcap_{\zeta>0,\varepsilon>0} \Gamma(\zeta,\varepsilon)$ and $\mu(\Gamma(\zeta,\varepsilon)) \to 0^+$ as $(\zeta,\varepsilon) \to (0^+,0^+)$. By the properties of μ mentioned in section 3, we see that $\boldsymbol{S}$ is compact and $H(\Gamma(\zeta,\varepsilon),\boldsymbol{S}) \to 0^+$ as $(\zeta,\varepsilon) \to (0^+,0^+)$.

Let $\{\boldsymbol{x}_n\} := \{(x_n,\lambda_n)\}$ be an approximating sequence. There is $\{(\zeta_n,\varepsilon_n)\} \to (0^+,0^+)$ such that, for all $\boldsymbol{y} \in \mathrm{gr}S$ and all $y_n \in K_2(x_n,\lambda_n)$,

$$F(\boldsymbol{x}_n,\boldsymbol{y}) + \varepsilon_n \geq 0,$$

$$f(x_n,y,\lambda_n) + \zeta_n \geq 0.$$

Therefore $(x_n,\lambda_n) \in \Gamma(\zeta_n,\varepsilon_n)$. Consequently,

$$d(x_n,\boldsymbol{S}) \leq H(\Gamma(\zeta_n,\varepsilon_n),\boldsymbol{S}) \to 0^+.$$

By the compactness of $\boldsymbol{S}$, there is a subsequence of $\boldsymbol{x}_n$ convergent to some point of $\boldsymbol{S}$. Hence, (EPEC) is well-posed. □

The following examples show that all assumptions of Theorem 5.3 (ii) are essential.

Example 5.6 (the closedness of K_1 is essential). Let $X = [-1,1], \Lambda = [0,1]$, $K_1(x,\lambda) = (-\lambda,1]$, $K_2(x,\lambda) \equiv [0,1]$, $f(x,y,\lambda) = x(x-y)$ and $F((x,\lambda_1),(y,\lambda_2)) = 2^{x+y}$. Then X is complete and

K_2 is lsc in $X \times \Lambda$. Assumptions (ii b) and (iii b) are fulfilled since f and F are continuous in $X \times X \times \Lambda$ and $(X \times \Lambda) \times (X \times \Lambda)$, respectively. Moreover, $\Gamma(\zeta, \varepsilon) \subseteq [-1,1] \times [0,1]$ and hence $\gamma(\Gamma(\zeta, \varepsilon)) \leq \gamma([-1,1] \times [0,1]) = 0$. It is easy to see that the solution set of (EPEC) coincides with gr$S$. But $S(0) = \{1\}$ and $S(\lambda) = \{0,1\}$ for all $\lambda \in (0,1]$, i.e., gr$S = (1,0) \cup \{(k,\lambda) \mid k = 0,1; \lambda \in (0,1]\}$. With the same arguments as in Example 5.2, (EPEC) is not well-posed. The reason is that K_1 is not closed at $(0,0)$.

Example 5.7 (the lower semicontinuity of K_2 cannot be dropped). Let X and Λ be as in Example 5.6, $K_1(x,\lambda) = [0,1]$, $f(x,y,\lambda) = x + y$, $F((x,\lambda_1),(y,\lambda_2)) = 1$ and

$$K_2(x,\lambda) = \begin{cases} \{-1,0,1\}, & \text{if } \lambda = 0, \\ \{0,1\}, & \text{otherwise.} \end{cases}$$

Then X is complete, K_1 is closed in $X \times \Lambda$ and (ii b) and (iii b) hold. $\Gamma(\zeta,\varepsilon) \subseteq [0,1] \times [0,1]$ and hence $\gamma(\Gamma(\zeta,\varepsilon)) = 0$. Furthermore, the solution set of (EPEC) is gr$S$. But $S(0) = \{1\}$, $S(\lambda) = \{0,1\}$ for all $\lambda \in (0,1]$. Hence, gr$S = (1,0) \cup \{(k,\lambda) \mid k = 0,1; \lambda \in (0,1]\}$. Thus, (EPEC) is not well-posed. The reason is that K_2 is not lsc in $X \times \Lambda$.

Example 5.8 ((ii b) cannot be dispensed). Let X, Λ, K_1, F as in Example 5.7, $K_2(x,\lambda) = \{\lambda, 1+\lambda\}$ and

$$f(x,y,\lambda) = \begin{cases} -1, & \text{if } x + y = 1, \\ 1, & \text{otherwise.} \end{cases}$$

Then X is complete, (ii a) and (ii c) are satisfied and $\gamma(\Gamma(\zeta,\varepsilon)) = 0$. But $S(0) = (0,1)$, $S(\lambda) = [0,1]$ for all $\lambda \in (0,1]$, i.e. gr$S = \big((0,1) \times \{0\}\big) \cup \{[0,1] \times \{\lambda\} \mid \lambda \in (0,1]\}$. Therefore, (EPEC) is not well-posed. Indeed, $\boldsymbol{x}_n = (0, \frac{1}{n})$ is a solution of (EPEC). We see that $\boldsymbol{x}_n$ tends to $\boldsymbol{x} = (0,0)$, but $\boldsymbol{x}$ does not belong to the solution set of (EPEC). The reason is that assumption (iib) is violated as shown in Example 3.8.

Example 5.9 ((ii c) is essential). Let X, Λ, K_1, K_2 be as in Example 5.8, $f(x,y,\lambda) = 1$ and

$$F((x,\lambda_1),(y,\lambda_2)) = \begin{cases} -1, & \text{if } x + y = 1, \lambda_1 = 0, \\ 1, & \text{otherwise.} \end{cases}$$

We see that X is complete, (iia) and (iib) hold and $\gamma(\Gamma(\zeta,\varepsilon)) = 0$. Clearly $S(\lambda) = [0,1]$ for all $\lambda \in [0,1]$. It is easy to see that the solution set of (EPEC) is the following subset of gr$S$: $\boldsymbol{S} = \big((0,1) \times \{0\}\big) \cup \{[0,1] \times \{\lambda\} \mid \lambda \in (0,1]\}$. By the same argument as in Example 5.8, one sees that (EPEC) is not well-posed. The reason is that assumption (iic) is violated.

6 Optimization problem with equilibrium constraints (OPEC).

We prove first a sufficient condition for well-posedness in topological settings.

THEOREM 6.1 *Assume that X is compact and*

(i) *in* $X \times \Lambda$, K_1 *is closed and* K_2 *is lsc;*

(ii) f *is upper 0-level closed wrt* id *in* $K_1(X, \Lambda) \times K_2(X, \Lambda) \times \Lambda \times \{0\}$;

(iii) g *is lower pseudocontinuous in* $X \times \Lambda$.

Then (OPEC) *is well-posed. Furthermore, if* $S(\lambda)$ *is a singleton, for all* $\lambda \in \Lambda$, *and* (OPEC) *possesses a unique solution, then this problem is uniquely well-posed.*

Proof. Set $F((x, \lambda_1), (y, \lambda_2)) = g(y, \lambda_2) - g(x, \lambda_1)$. To apply Theorem 5.1, we need to check only that $F(., \boldsymbol{y})$ is upper 0-level closed wrt id in $X \times \Lambda$ for all $\boldsymbol{y} \in X \times \Lambda$. Let $\boldsymbol{x}_n = (x_n, \lambda_n) \in X \times \Lambda$ and $\varepsilon_n \in (0, +\infty)$ be such that $\{(x_n, \lambda_n, \varepsilon_n)\} \to (x, \bar{\lambda}, 0)$ and

$$F(\boldsymbol{x}_n, \boldsymbol{y}) + \varepsilon_n \geq 0.$$

Suppose

$$g(y, \lambda) < g(x, \bar{\lambda}).$$

Since g is lower pseudocontinuous at $(x, \bar{\lambda})$, one has

$$g(y, \lambda) < \liminf g(x_n, \lambda_n).$$

So, there are $t \in R$ and $n_0 \in \mathcal{N}$ such that, for all $n \geq n_0$,

$$g(y, \lambda) - g(x_n, \lambda_n) \leq g(y, \lambda) - t < 0.$$

This is impossible since $g(y, \lambda) - g(x_n, \lambda_n) + \varepsilon_n \geq 0$ for all n. The assertion on unique well-posedness is easy to be demonstrated. □

Remark 6.1

(i) In the special case where $K_1(x, \lambda) \equiv K_2(x, \lambda) = X$, it is easy to see that in (ii) we can replace f by "$f(., y, .)$, for all $y \in X$". So the unique well-posedness part of Theorem 6.1 improves Theorem 4.5 of [12]. Indeed, by using similar arguments as in Remark 3.2, the lower semicontinuity of $f(x, ., .)$ together with the monotonicity of $f(., ., \lambda)$ imply condition (ii) of Theorem 6.1. The lower semicontinuity of g required in Theorem 4.5 of [12] is more restrictive than our assumption (iii). Moreover, we omit the hemicontinuity of $f(., ., \lambda)$ and convexity of $f(x, ., .)$ and g. Note further that the lower pseudocontinuity of g imposed in Theorem 6.1 is strictly weaker than the lower semicontinuity requirement in Theorem 4.5 of [12].

(ii) In [22] an optimization problem with variational inequality constraints, a special case of our (OPEC), was investigated for X being a reflexive Banach space. Well-posedness and unique well-posedness results were established under the assumption that the parametric problem defining the constraint is parametrically well-posed or parametrically uniquely well-posed, respectively. In our results here, we impose only explicit conditions on the data of the problems. However, it is not hard to modify Theorem 6.1 to include properly the counterparts in [22].

For $\zeta, \varepsilon > 0$, the approximate solution set of (OPEC) is defined by

$$M(\zeta,\varepsilon) = \{(x,\lambda) \in K_1(x,\lambda) \times \Lambda \mid g(x,\lambda) \leq \inf_{\hat{\lambda}\in\Lambda, y\in S(\hat{\lambda})} g(y,\hat{\lambda}) + \varepsilon \text{ and}$$

$$f(x,y,\lambda) + \zeta \geq 0, \forall y \in K_2(x,\lambda)\}.$$

Theorem 6.2 *Let X and Λ be metric spaces. Then the following assertions hold.*

(i) $\mathrm{diam} M(\zeta,\varepsilon) \to 0^+$ *as* $(\zeta,\varepsilon) \to (0^+,0^+)$*, if* (OPEC) *is uniquely well-posed.*

(ii) *For the converse assume that X and Λ are complete and that the following conditions hold*

(a) *in $X \times \Lambda$, K_1 is closed and K_2 is lsc;*

(b) *f is upper 0-level closed wrt* id *in $K_1(X,\Lambda) \times K_2(X,\Lambda) \times \Lambda \times \{0\}$;*

(c) *g is lower pseudocontinuous in $X \times \Lambda$.*

Then (OPEC) *is uniquely well-posed, provided that* $\mathrm{diam} M(\zeta,\varepsilon) \to 0^+$ *as* $(\zeta,\varepsilon) \to (0^+,0^+)$.

Proof. (i) Assume that (OPEC) is uniquely well-posed. Arguing ad absurdum suppose the existence of $\{(\zeta_n,\varepsilon_n)\} \to (0^+,0^+)$, $n_0 \in \mathcal{N}$ and $r > 0$ such that, for all $n \geq n_0$, $\mathrm{diam} M(\zeta_n,\varepsilon_n) > r$. Then, (x_n^1,λ_n^1) and (x_n^2,λ_n^2) in $M(\zeta_n,\varepsilon_n)$ exist such that $d((x_n^1,\lambda_n^1),(x_n^2,\lambda_n^2)) > \frac{r}{2}$. Being approximating sequences, $\{(x_n^1,\lambda_n^1)\}$ and $\{(x_n^2,\lambda_n^2)\}$ converge to the unique solution and we get the contradiction

$$\lim_{n\to+\infty} d((x_n^1,\lambda_n^1),(x_n^2,\lambda_n^2)) = 0.$$

(ii) Assume that $\{\boldsymbol{x}_n\} := \{(x_n,\lambda_n)\}$ is an approximating sequence for (OPEC). Then there is some $\{(\zeta_n,\varepsilon_n)\} \to (0^+,0^+)$ such that

$$g(x_n,\lambda_n) \leq \inf_{\hat{\lambda}\in\Lambda, y\in S(\hat{\lambda})} g(y,\hat{\lambda}) + \varepsilon_n,$$

$$f(x_n,y,\lambda_n) + \zeta_n \geq 0, \forall y \in K_2(x_n,\lambda_n).$$

This means that $\boldsymbol{x}_n = (x_n,\lambda_n) \in M(\zeta_n,\varepsilon_n)$ and hence $\{\boldsymbol{x}_n\}$ is a Cauchy sequence and converges to some point $\bar{\boldsymbol{x}} = (\bar{x},\bar{\lambda})$. Since K_1 is closed at $(\bar{x},\bar{\lambda})$ and $x_n \in K_1(x_n,\lambda_n)$, one has $\bar{x} \in K_1(\bar{x},\bar{\lambda})$. Using the same argument as for Theorem 6.1, one sees that $\bar{\boldsymbol{x}}$ solves (OPEC). It remains to show that (OPEC) has a unique solution. If (OPEC) has two distinct solutions $(\bar{x}_1,\bar{\lambda}_1)$ and $(\bar{x}_2,\bar{\lambda}_2)$, they must belong to $M(\zeta,\varepsilon)$, for all $\zeta,\varepsilon > 0$. This yields a contradiction that

$$0 < d((\bar{x}_1,\bar{\lambda}_1),(\bar{x}_2,\bar{\lambda}_2)) \leq \mathrm{diam} M(\zeta,\varepsilon). \quad \square$$

For well-posedness of (OPEC) in terms of measures of noncompactness we have the following result.

Theorem 6.3

(i) *If* (OPEC) *is well-posed, then* $\gamma(M(\zeta,\varepsilon)) \to 0^+$ *as* $(\zeta,\varepsilon) \to (0^+,0^+)$*, where* γ *is either the Kuratowski or the Hausdorff measure of noncompactness.*

(ii) *Conversely, if* X *and* Λ *are complete and if the following conditions hold*

(a) *in* $X \times \mathit{\Lambda}$*,* K_1 *is closed and* K_2 *is lsc;*

(b) f *is upper* b*-upper level closed in* $K_1(X,\mathit{\Lambda}) \times K_2(X,\mathit{\Lambda}) \times \mathit{\Lambda}$*, for all* $b < 0$;

(c) g *is lsc in* $X \times \mathit{\Lambda}$*,*

then (OPEC) *is well-posed, provided that* $M(\Gamma(\zeta,\varepsilon)) \to 0^+$ *as* $(\zeta,\varepsilon) \to (0^+,0^+)$.

Proof. Let us consider only the case of the Hausdorff measure $\gamma = \eta$.

(i) Assume that (OPEC) is well-posed. For all $\zeta,\varepsilon > 0$, the solution set $\boldsymbol{S}$ of (OPEC) satisfies obviously the containment $\boldsymbol{S} \subseteq M(\zeta,\varepsilon)$. Hence,

$$H(M(\zeta,\varepsilon),\boldsymbol{S}) = H^*(M(\zeta,\varepsilon),\boldsymbol{S}).$$

Any sequence $\{\boldsymbol{x}_n\}$ in $\boldsymbol{S}$ is an approximating sequence of (OPEC) and has a subsequence convergent to some point of $\boldsymbol{S}$. So, $\boldsymbol{S}$ is compact.

Let $\boldsymbol{S} \subseteq \bigcup_{k=1}^{n} B(z_k,\varepsilon)$. Then $M(\zeta,\varepsilon) \subseteq \bigcup_{k=1}^{n} B\big(z_k,\varepsilon + H(M(\zeta,\varepsilon),\boldsymbol{S}\big)$. Therefore,

$$\eta(M(\zeta,\varepsilon)) \le H(M(\zeta,\varepsilon),\boldsymbol{S}) + \eta(\boldsymbol{S}) = H(M(\zeta,\varepsilon),\boldsymbol{S}).$$

To prove that $H(M(\zeta,\varepsilon),\boldsymbol{S}) \to 0^+$ as $(\zeta,\varepsilon) \to (0^+,0^+)$ by contradiction suppose the existence of $\rho > 0$, $\{(\zeta_n,\varepsilon_n))\} \to (0^+,0^+)$ and $\boldsymbol{x}_n \in M(\zeta_n,\varepsilon_n)$ such that, for all $n \in \mathcal{N}$,

$$d(\boldsymbol{x}_n,\boldsymbol{S}) \ge \rho.$$

Being an approximating sequence for (EPEC), $\{\boldsymbol{x}_n\}$ has a subsequence convergent to some point of $\boldsymbol{S}$, which is impossible.

(ii) Assume that $\eta(M(\zeta,\varepsilon)) \to 0^+$ as $(\zeta,\varepsilon) \to (0^+,0^+)$. We first check that $M(\zeta,\varepsilon)$ is closed for all $\zeta,\varepsilon > 0$. Let $\boldsymbol{x}_n := (x_n\lambda_n) \in M(\zeta,\varepsilon)$ with $\{\boldsymbol{x}_n\} \to \boldsymbol{x} := (x,\lambda)$. Hence,

$$g(x_n,\lambda_n) \le \inf_{\hat{\lambda}\in\Lambda, y\in S(\hat{\lambda})} g(y,\hat{\lambda}) + \varepsilon,$$

$$f(x_n,y,\lambda_n) + \zeta \ge 0, \forall y \in K_2(x_n,\lambda_n).$$

Since K_1 is closed at (x,λ), $x \in K_1(x,\lambda)$. By the semicontinuity of g at (x,λ), we have

$$g(x,\lambda) \le \liminf g(x_n,\lambda_n) \le \inf_{\hat{\lambda}\in\Lambda, y\in S(\hat{\lambda})} g(y,\hat{\lambda}) + \varepsilon.$$

Furthermore, we claim that, for all $y \in K_2(x, \lambda)$,

$$f(x, y, \lambda) + \zeta \geq 0.$$

Indeed, if there exists $y \in K_2(x, \lambda)$ such that $f(x, y, \lambda) + \zeta < 0$, there is $y_n \in K_2(x_n, \lambda_n)$ such that $y_n \to y$, as K_2 is lsc at (x, λ). By the upper $-\zeta$-level closedness of f at (x, y, λ), there is $n_0 \in \mathcal{N}$ such that, for all $n \geq n_0$,

$$f(x_n, y_n, \lambda_n) < -\zeta,$$

which is a contradiction. Hence, $M(\zeta, \varepsilon)$ is closed. Note further that $\boldsymbol{S} = \bigcap_{\zeta>0,\varepsilon>0} M(\zeta, \varepsilon)$ and $\eta(M(\zeta, \varepsilon)) \to 0^+$ as $(\zeta, \varepsilon) \to (0^+, 0^+)$. Therefore, by the earlier-mentioned properties of η, $\boldsymbol{S}$ is compact and $H(M(\zeta, \varepsilon), \boldsymbol{S}) \to 0^+$ as $(\zeta, \varepsilon) \to (0^+, 0^+)$.

Let $\{\boldsymbol{x}_n\} := \{(x_n, \lambda_n)\}$ be an approximating sequence, i.e. there exists $\{(\zeta_n, \varepsilon_n)\} \to (0^+, 0^+)$ such that

$$g(x_n, \lambda_n) \leq \inf_{\hat{\lambda} \in \Lambda, y \in S(\hat{\lambda})} g(y, \hat{\lambda}) + \varepsilon_n,$$

$$f(x_n, y, \lambda_n) + \zeta_n \geq 0, \forall y \in K_2(x_n, \lambda_n).$$

Consequently, $(x_n, \lambda_n) \in M(\zeta_n, \varepsilon_n)$. So,

$$d(x_n, \boldsymbol{S}) \leq H(M(\zeta_n, \varepsilon_n), \boldsymbol{S}) \to 0^+.$$

By the compactness of $\boldsymbol{S}$, there is a subsequence of $\{\boldsymbol{x}_n\}$ convergent to some point of $\boldsymbol{S}$. Thus, (OPEC) is well-posed. □

Remark 6.2 For the special case mentioned in Remark 6.1 (i), Theorems 4.1 and 4.2 of [12] contain similar results for the case $\gamma = \mu$. Theorem 6.2 improves Theorem 4.1 since we omit the assumptions encountered in Remark 6.1. Our Theorem 6.3 is an improvement of Theorem 4.2 of [12], as the lower pseudocontinuity of g in (iic) is weaker than the lower semicontinuity imposed there.

Acknowledgments This work was supported by the National Foundation for Science and Technology Development of Vietnam.

References

[1] L. Q. Anh and P. Q. Khanh, *Semicontinuity of the solution sets of parametric multivalued vector quasiequilibrium problems*, J. Math. Anal. Appl., 294 (2004), pp. 699-711.

[2] L. Q. Anh and P. Q. Khanh, *On the stability of the solution sets of general multivalued vector quasiequilibrium problems*, J. Optim. Theory Appl., 135 (2007), pp. 271-284.

[3] L. Q. Anh and P. Q. Khanh, *Various kinds of semicontinuity and the solutions ets of parametric multivalued symmetric vector quasiequilibrium problems*, J. Glob. Optim., 41 (2008), pp. 539-558.

[4] L. Q. Anh and P. Q. Khanh, *Semicontinuity of the approximate solution sets of multivalued quasiequilibrium problems*, Numer. Funct. Anal. Optim., 29 (2008), pp. 24-42.

[5] L. Q. Anh, P. Q. Khanh, D. T. M. Van and J. C. Yao, *Well-posedness for vector quasiequilibria*, Taiwan. J. Math., 13 (2009), pp. 713-737.

[6] J. Banas and K. Goebel, *Measures of Noncompactness in Banach Spaces,* Lecture Notes in Pure and Applied Mathematics 60, Marcel Dekker, New York-Basel, 1980.

[7] G. Beer and R. Lucchetti, *The epi-distance topology,* Math. Oper. Res., 17 (1992), pp. 715-726.

[8] L. C. Ceng, N. Hadjisavvas, S. Schaible and J. C. Yao, *Well-posedness for mixed quasivariational-like inequalities*, J. Optim. Theory Appl., 139 (2008), pp. 109-225.

[9] G. P. Crespi, A. Guerraggio and M. Rocca, *Well-posedness in vector optimization problems and vector variational inequalities*, J. Optim. Theory Appl., 132 (2007), pp. 213-226.

[10] J. Danes, *On the Istratescu measure of noncompactness*, Bull. Math. Soc. R.S. Roumanie, 16 (1972), pp. 403-406.

[11] Y. P. Fang, N. J. Huang and J. C. Yao, *Well-posedness of mixed variational inequalities, inclusion problems and fixed point problems*, J. Glob. Optim., 41 (2008), pp. 117-133.

[12] Y. P. Fang, R. Hu and N.J. Huang, *Well-posedness for equilibrium problems and for optimization problems with equilibrium constraints*, Computers Math. Appl., 55 (2008), pp. 89-100.

[13] J. Hadamard, *Sur le problèmes aux dérivees partielles et leur signification physique*, Bull. Univ. Princeton, 13 (1902), pp. 49-52.

[14] X. X. Huang and X. Q. Yang, *Gereralized Levitin-Polyak well-posedness in constrained optimization*, SIAM J. Optim., 17 (2006), pp. 243-258.

[15] A. Ioffe and R. E. Lucchetti, *Typical convex program is very well-posed*, Math. Prog., Ser. B, 104 (2005), pp. 483-499.

[16] A. Ioffe, R. E. Lucchetti and J. P. Revalski, *Almost every convex or quadratic programming problem is well-posed*, Math. Oper. Res., 29 (2004), pp. 369-382.

[17] K. Kimura, Y. C. Liou, S. Y. Wu and J. C. Yao, *Well-posedness for parametric vector equilibbrium problems with applications*, J. of Indust. Manag. Optim., 2009, to appear.

[18] B. Lemaire, C. Ould Ahmed Salem and J. P. Revalski, *Well-posedness of variational problems with applications to staircase methods*, C. R. Acad. Sci. Paris 332, Ser. 1 (2001), pp. 943-948.

[19] B. Lemaire, C. Ould Ahmed Salem and J. P. Revalski, *Well-posedness by perturbations of variational problems*, J. Optim. Theory Appl., 115 (2002), pp. 345-368.

[20] E. S. Levitin and B. T. Polyak, *Convergence of minimizing sequences in conditional extremum problems*, Soviet Math. Dokl., 7 (1966), pp. 764-767.

[21] M. B. Lignola, *Well-posedness and L-well-posedness for quasivariational inequalities*, J. Optim. Theory Appl., 128 (2006), pp. 119-138.

[22] M. B. Lignola and J. Morgan, *Well-posedness for optimization problems with constraints defined by variational inequalities having a unique solution*, J. Glob. Optim., 16 (2000), pp. 57-67.

[23] M. B. Lignola and J. Morgan, *α-Well-posedness for Nash equilibria and for optimization problems with Nash equilibrium constraints*, J. Glob. Optim., 36 (2006), pp. 439-459.

[24] R. Lucchetti and F. Patrone, *Hadamard and Tikhonov well-posedness of a certain class of convex functions*, J. Math. Anal. Appl., 88 (1982), pp. 204-215.

[25] M. Margiocco, F. Patrone and L. Pusillo Chicco, *A new approach to Tikhonov well-posedness for Nash equilibria*, Optim., 40 (1997), pp. 385-400.

[26] J. Morgan and V. Scalzo, *Pseudocontinuity in optimization and nonzero sum games*, J. Optim. Theory Appl., 120 (2004), 181-197.

[27] J. Morgan and V. Scalzo, *Discontinuous but well-posed optimization problems*, SIAM J. Optim., 17 (2006), pp. 861-870.

[28] V. Rakočěvíc, *Measures of noncompactnaess and some applications*, Filomat, 12 (1998), pp. 87-120.

[29] J. P. Revalski, *Hadamard and strong well-posedness for convex programs*, SIAM J. Optim., 7 (1997), pp. 519-526.

[30] A. N. Tikhonov, *On the stability of the functional optimization problem*, Soviet Comput. Math. Math. Phys., 6 (1966), pp. 28-33.

[31] T. Zolezzi, *Well-posedness criteria in optimization with applications to the calculus of variations*, Nonlinear Anal. TMA., 25 (1995), pp. 437-453.

[32] T. Zolezzi, *Well-posedness and optimization under perturbations*, Annals Oper. Res., 101 (2001), pp. 351-361.

[33] T. Zolezzi, *Condition number theorems in optimization*, SIAM J. Optim., 14 (2003), pp. 507-516.

[34] T. Zolezzi, *On well-posedness and conditioning in optimization*, ZAMM Z. Angew. Math. Mech., 84 (2004), pp. 435-443.